

MONOMIALIZATION OF SINGULAR METRICS ON REAL SURFACES

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ABSTRACT. Let B be a real analytic vector bundle of rank 2 over a smooth real analytic surface S , equipped with a real analytic fiber-metric \mathbf{g} and such that there exists a real analytic mapping of vector bundles $TS \rightarrow B$ inducing an isomorphism outside a proper sub-variety of S . Let κ be a real analytic 2-symmetric tensor field on B . Our main result, Theorem 9.2, roughly states the following: There exists a locally finite composition of points blowings-up $\sigma : \tilde{S} \rightarrow S$ such that there exists a unique pair of real analytic singular foliations \mathcal{F}_1 and \mathcal{F}_2 on \tilde{S} - only with simple singularities adapted to the exceptional divisor \tilde{E} and orthogonal for the (regular extension of the) pull back on \tilde{S} of the fiber-metric \mathbf{g} (only semi-positive definite along \tilde{E}) - locally simultaneously diagonalizing the pull-back on \tilde{S} of the original 2-symmetric tensor field κ .

When S is the resolved surface of an embedded resolution of singularities of an embedded real analytic surface singularity S_0 our result thus yields a local presentation of the extension $\tilde{\mathbf{h}}$ of the pull-back on \tilde{S} of the inner-metric of S_0 at any point of the exceptional divisor \tilde{E} . We furthermore recover that the pulled-back inner metric $\tilde{\mathbf{h}}$ is locally quasi-isometric to the sum of the (symmetric tensor) square of differentials of (independent) monomials in the exceptional divisor \tilde{E} , namely, *Hsiang & Pati property* is satisfied at every point of the resolved surface \tilde{S} .

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1. INTRODUCTION AND STATEMENT OF A SIMPLER VERSION OF THE RESULT

Any smooth 2-symmetric tensor over a smooth Riemannian manifold M is point-wise diagonalizable. In general the collection of this diagonalizing bases do not form a local orthogonal frame everywhere on M . This defect of local simultaneous diagonalization may become a serious inconvenience for practical purpose.

Key words and phrases. singular metrics; singular surfaces; resolution of singularities; singular foliations; simple singularities of foliations; Hsiang & Pati property.

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Kurdyka & Paunescu proved a special parameterization Theorem [18] for real analytic families $(A(u))_{u \in \mathcal{U}}$ of $n \times n$ -symmetric matrices over an open subset \mathcal{U} of an Euclidean space \mathbb{R}^p , using a real analytic and surjective mapping $\pi : \mathcal{V} \rightarrow \mathcal{U}$ obtained as a finite composition of geometrically admissible blowings-up, which allow these authors to (among other things) locally simultaneously diagonalize everywhere on \mathcal{V} the pulled-back family $(A \circ \pi(v))_{v \in \mathcal{V}}$.

Our primary motivation related to this topic comes from trying to locally describe, in a simple fashion, real analytic 2-symmetric tensors over a real analytic Riemannian manifold which may degenerate somewhere, as well as the restriction of real analytic 2-symmetric tensors over a real analytic Riemannian (or Hermitian) ambient manifold M to (the regular part of) a given real (or complex) analytic singular sub-variety S .

An important example, for applications, of such a situation is when the 2-symmetric tensor is the ambient metric itself, providing on the analytic set S , for which the distance is taken in M .

Let us say a bit more about the inner metric stakes. The problem is understanding how the inner Riemannian (Hermitian) structure on the regular part of the singular sub-variety accumulates at the singular part. An accepted scheme, which hopefully will help to provide a local description of the inner metric nearby the singular locus, is to parameterize the (embedded) singular sub-variety by a regular manifold by means of a (bi-rational like) regular surjective mapping (the resolution mapping of a desingularization of the singular sub-variety). Then we pull back the inner metric onto the resolved manifold which becomes a regular semi-Riemannian metric, and control what happens on the preimage of the singular locus of the original sub-variety (the exceptional locus of the resolution) since the loss of positive-definiteness can only occur there. Further *carefully chosen* blowings-up should yield a better description of the iterated pull-back of the metric. The meaning of *carefully chosen* centers of blowings-up is yet to be systematically developed in regard of the control that can be guaranteed after pull-back.

Hsiang and Pati were the first to provide a local description of inner metric of normal complex surface singularity germs [17, Sections II & III] along the proposed scheme. Later Pardon and Stern presented a more conceptual point of view of this result [20, Section 3]. Grieser had also found a real analytic version counterpart of this result, though it is unpublished [12].

Almost fifteen years after Hsiang & Pati, Youssin announced that such a local description exists on some resolved manifold of any given complex algebraic singularity [24]. The short version can be formulated as follows:

Youssin Conjecture [24, 2]: *Given a singular complex algebraic sub-variety X_0 of pure dimension n embedded in a complex (algebraic) manifold M_0 , there exists a resolution of singularities of $\sigma : (X, E) \rightarrow X_0$, which is a locally finite composition of blowings-up with regular (algebraic) centers, such that at each point \underline{a} of the exceptional divisor E , the pull-back, onto the resolved manifold X , of the inner metric of X_0 (inherited from the ambient Hermitian metric in M_0) by the resolution mapping σ is (locally) quasi-isometric to a sum of the "Hermitian squares" of (exactly n independent) differentials of monomials in the exceptional divisor E (with additional properties on the integral powers).*

A few articles claiming proving the complex 3-dimensional case of Youssin's conjecture were published in the late 1990s and early 2000s, but they were either incomplete or with serious errors. Nevertheless the very recent preprint [2] proposes a problem equivalent to Youssin's which is better adapted to resolution singularities techniques. Moreover this preprint closes Youssin's Conjecture in the case of real and complex surfaces (algebraic or analytic) and also provides a proof for 3-folds singularities.

Our goal in the present paper is to focus only on the case of real analytic surface singularities from the point of view we started with in the introduction. Moreover our previous joint works [10], [11] dealing with the inner metric of singular surfaces and [9] dealing with a singular metric on a regular surface, are exemplifying the need of a description of the inner metric which is finer than Hsiang & Pati's.

Given a real analytic surface singularity S in an ambient real analytic manifold M and given a 2 symmetric tensor κ on M , we would like to know, as in the case of the inner metric, whether we may obtain a useful and simple presentation of the pull-back, on a resolved surface, of the restriction of κ onto S .

The problems presented above whence considered from the point of view of a resolved manifold, is a special occurrence of the following general situation: Let X_1 be a regular (i.e. real or complex algebraic or analytic) manifold and let (B_1, \mathbf{g}_1) be a regular vector bundle over X_1 of finite rank and equipped with a regular fiber-metric (Riemannian or Hermitian) \mathbf{g}_1 . Let κ_1 be a 2-symmetric regular tensor field on B_1 , (later shortened as 2-symmetric tensor on B_1). As already suggested some features of κ can be investigated following two similar but non-equivalent approaches:

Parameterization Problem. *Finding a smooth and surjective mapping $\sigma_2 : X_2 \rightarrow X_1$, and describe the features of $\kappa_1 \circ \sigma_2$ as objects with regular variations on X_2 . Note that $\kappa_1 \circ \sigma_2$ is a 2-symmetric tensor on the regular vector bundle $\sigma_2^* B_1$.*

Resolution Problem. *When there exists a regular mapping of vector bundles $TX_1 \rightarrow B_1$ which is an isomorphism outside a proper sub-variety of X_1 , find a regular and surjective mapping $\sigma_2 : X_2 \rightarrow X_1$ such that the features of the pulled-back of the 2-symmetric tensor κ_1 , now a 2-symmetric tensor on X_2 , are as good as can be.*

The Parameterization Problem consists mostly of regularizing some functions (roots, components of vector fields,...). Kurdyka & Paunescu's results (generalizing Rellich's complex one dimensional case) provide almost immediately an answer to the Parameterization Problem when X_1 is an open subset of an Euclidean space.

The Resolution Problem as far as we know has never been solved (or even addressed in this form). It is significantly harder than the associated Parameterization Problem. Nevertheless in practice, without saying so we, we very likely have to start with solving this latter one.

The present paper provides an answer to the Resolution Problem for a "singular" 2-symmetric tensor of a real analytic vector bundle of rank two over a regular real analytic surface which is an almost tangent bundle (see Definition 9.1). In particular it applies (after some preliminary preparations) to the inner metric of a real analytic surface singularity embedded in real analytic Riemannian manifold.

We present below statements of our results for the inner metric of a real analytic singular surface since it is the simplest situation we can encounter.

Let X_0 be a real analytic surface singularity of a real analytic Riemannian manifold (M_0, \mathbf{g}_0) , supporting a real analytic space structure $(X_0, \mathcal{O}_{X_0} := \mathcal{O}_{M_0}/I_0)$, for a coherent \mathcal{O}_{M_0} -ideal sheaf I_0 . Let Y_0 be the (non-empty) singular locus of X_0 . Let \mathcal{N}_{X_0} be the closure of $T(X_0 \setminus Y_0)$ taken in TM_0 .

Our first result (see Proposition 4.5 for the general case) is a parameterization result of a global simultaneous diagonalization, namely:

Proposition 1.1. *There exists $\sigma_1 : (X_1, E_1) \rightarrow (X_0, Y_0)$, a locally finite composition of geometrically admissible blowings-up, such that X_1 is smooth, E_1 is a normal crossing divisor and $B_1 := \sigma_1^*(\mathcal{N}_{X_0})$ is a real analytic vector bundle over X_1 . Moreover, there exists, up to permutation, a unique pair of \mathcal{O}_{X_1} -invertible sub-modules \mathcal{L}_1 and \mathcal{L}_2 of $\Gamma_{X_1}(B_1^*)$ both nowhere vanishing in X_1 , such that for every point \underline{a}_1 of X_1 there exists an open neighborhood \mathcal{U}_1 of \underline{a}_1 such that*

i) *If θ_i is a local generator of \mathcal{L}_i at \underline{a}_1 , then for each $\underline{a} \in \mathcal{U}_1$, the vector lines $\ker \theta_1(\underline{a})$ and $\ker \theta_2(\underline{a})$ of $T_{\sigma_1(\underline{a})} M_0$ intersect orthogonally;*

ii) *If $\sigma_1(\underline{a})$ is not a singular point of X_0 , then $\ker \theta_i(\underline{a})$ is contained in $T_{\sigma_1(\underline{a})} X_0$;*

iii) *The metric parameterized by σ_1 writes on \mathcal{U}_1 as*

$$(\mathbf{g}_0|_{\mathcal{N}_{X_0}}) \circ \sigma_1 = (\mathcal{M}_1 \cdot \theta_1) \otimes (\mathcal{M}_1 \cdot \theta_1) + (\mathcal{M}_2 \cdot \theta_2) \otimes (\mathcal{M}_2 \cdot \theta_2) = \mathcal{M}_1^2 \theta_1 \otimes \theta_1 + \mathcal{M}_2^2 \theta_2 \otimes \theta_2;$$

where \mathcal{M}_i is a monomial in E_1 , and $i = 1, 2$

iv) *Some further technical properties.*

More precisely, there exists a global regular orthogonal directional frame on the regular vector bundle $B_1 = \sigma_1^*(\mathcal{N}_{X_0})$ over X_1 , in which $(g_0|_{X_0}) \circ \sigma_1$ is locally simultaneously diagonalized so that the size of the (local) generators of the diagonalizing frame are monomials times a local unit (compare with [18]). It is also stable under further points blowings-up. From here, we get the *monomialization* of the pulled-back metric, leading to Theorem 9.2 presented in the following simpler form:

Theorem 1.2. *There exists a locally finite composition of geometrically admissible blowings-up*

$$\sigma_2 : (X_2, E_2) \xrightarrow{\beta_2} (X_1, E_1) \xrightarrow{\sigma_1} (X_0, Y_0)$$

such that there exists, up to permutation, a pair of \mathcal{O}_{X_2} -invertible sub-modules Θ_1 and Θ_2 of $\Omega_{X_2}^1$, such that

- i) Each foliation Θ_i admits only simple singularities adapted to E_2 .*
- ii) Every point \underline{a}_2 of X_2 admits an open neighborhood $\mathcal{U}_2 \ni \underline{a}_2$ such that if $\sigma_2(\underline{a}_2)$ is not a singular point of X_0 , then $(D\sigma_2(\underline{a}_2))(\ker \Theta_1(\underline{a}_2))$ and $(D\sigma_2(\underline{a}_2))(\ker \Theta_2(\underline{a}_2))$ are orthogonal lines of $T_{\sigma_2(\underline{a}_2)}X_0$;*
- iii) The pull-back of the metric $g_0|_{X_0}$ by the resolution mapping σ_2 extends on X_2 as a real analytic semi-Riemannian metric g_2 , which writes on \mathcal{U}_2 as*

$$g_2 = (\mathcal{M}_1 \cdot \mu_1) \otimes (\mathcal{M}_1 \cdot \mu_1) + (\mathcal{M}_2 \cdot \mu_2) \otimes (\mathcal{M}_2 \cdot \mu_2) = \mathcal{M}_1^2 \mu_1 \otimes \mu_1 + \mathcal{M}_2^2 \mu_2 \otimes \mu_2$$

where μ_i is a local generator of Θ_i on \mathcal{U}_2 and where \mathcal{M}_i is a monomial in E_2 , and $i = 1, 2$

- iv) Some further technical properties.*

Of course what is also important is point iv) of the main theorem, since we unexpectedly (that is without further blowings-up) obtain the following (Corollary 10.9 and Corollary 10.12):

Corollary. *Under the hypotheses of Theorem 1.2 each point \underline{a}_2 in E_2 admits Hsiang & Pati coordinates, namely there exists local analytic coordinates (u, v) centered at \underline{a}_2 such that*

- i) If \underline{a}_2 is a smooth point of E_2 , there exists local coordinates (u, v) at \underline{a}_2 such that $(E_2, \underline{a}_2) = \{u = 0\}$, then the (extension of the) pulled-back metric g_2 is quasi-isometric (nearby \underline{a}_2) to the metric given by*

$$du^{k+1} \otimes du^{k+1} + d(u^{l+1}v) \otimes d(u^{l+1}v)$$

for non-negative integer numbers $l \geq k$.

- ii) If \underline{a}_2 is a corner point of E_2 , there exists local coordinates (u, v) at \underline{a}_2 such that $(E_2, \underline{a}_2) = \{uv = 0\}$, then the (extension of the) pulled-back metric g_2 is quasi-isometric (nearby \underline{a}_2) to the metric given by*

$$d(u^m v^n) \otimes d(u^m v^n) + d(u^k v^l) \otimes d(u^k v^l)$$

for positive integer numbers $m \leq k, n \leq l$ and such that $ml - kn \neq 0$.

The proofs of our results combine resolution of singularities of "varieties", with resolution of singularities of plane foliations and with some further tailored local computations of similar types. We follow simple geometric ideas. First, a bilinear symmetric form (or any non-zero multiple of it) on a finitely dimensional real vector space is a "sum of squares" in an appropriate orthogonal basis. Second, we are hoping to find a parameterization allowing to locally simultaneously diagonalize the 2-symmetric tensor locally (at any point) so that over any such a neighborhood it is a sum of the (symmetric tensor) squares (of regular differential 1-form germs). Third, pulling back everything onto the parameter space (here the resolved surface) we obtain two (likely singular) foliations in which the pull-back of the 2-symmetric tensor onto the parameter space is still a "sum of squares". The desingularization of plane foliations provides isolated singularities as simple as possible and thus yields a presentation of the pull-back of the 2-symmetric tensor as a sum of squares of two well described 1-forms. This scheme gives Theorem 9.2.

Further tailored work will control further data about the pair of foliations as well as about the pull-back of the 2-symmetric tensor. Eventually, when everything is as reasonably presented as it can be, the Hsiang & Pati property holds true in the case of the inner metric of a surface singularity. All this extra preparation is absolutely necessary to guarantee that property.

Since the statement of the main result is given over a regular surface, when starting from a singular subvariety, we carry our problem out onto a regular resolved surface via blowings-up. A key step in this process

is the existence of Gauss regular resolution of singularities (Definition 7.1) which allows to have a Parameterization Problem on the corresponding resolved surface, and thus to carry on towards solving the corresponding resolution problem.

This article is organized as follows:

Section 2 presents basic material needed throughout the paper and set some notations. We recall Theorem 2.1, namely Hironaka's resolution of singularities Theorem [15, 1] in a form best suited for our purpose.

In Section 3, we define a projective form associated with a real bilinear symmetric "form" over a real vector bundle, since it will prove more convenient to handle resolution of singularities techniques.

The parameterization problem strictly speaking is solved by Proposition 4.5 (presented above in a simplified form in a special case), which is the main result of Section 4.

Section 5 recalls what is a resolution of singularities of a real analytic plane singular foliation.

In Section 6, we investigate pairs of generically transverse plane singular foliations, give some technical results about the behavior of such a pair with respect to a prescribed normal crossing divisor. We recall the logarithmic point of view to present the resolution of singularities of a plane foliation [5, 6, 7]. We highlight the fact that what we develop in this section is absolutely essential for the Hsiang & Pati property.

Section 7 and Section 8 deal with the notion of the restriction of 2-symmetric tensor onto a singular sub-variety. To that end, we present (unable to find a reference) a proof of Proposition 7.2 stating (the well-known fact of) the existence of Gauss-regular resolution of singularities, namely, for which the pull-back of the Gauss map of the initial singular sub-variety, extends regularly, to the whole resolved manifold.

Our main result, Theorem 9.2 presented in Section 9 solves the resolution problem. The local normal form obtained for the pull-back of the initial 2-symmetric tensor is stable under further blowings-up. We need further definitions (e.g. *almost tangent bundle*) and introduce a fundamental change in the notations to distinguish the pull-back as a base change (composition on right by the resolution mapping) and the pull-back in the sense of differentials (composition on the right with the differential of the resolution mapping).

The unexpected consequence (when we started this work) of our long and detailed Theorem 9.2, in the case of inner metrics on singular surfaces, is described in Section 10. We discover, thanks to the normal forms of Section 6, that the geometric point of view we have chosen combined with all the extra properties we obtained already provides everywhere the Hsiang & Pati local forms.

2. SETTING - RESOLUTION OF SINGULARITIES THEOREMS

A *regular manifold* is a real analytic manifold. A *regular sub-manifold* of a given regular manifold is a real analytic sub-manifold. A *regular mapping* $M \rightarrow N$ is a real analytic mapping between regular manifolds M and N . A *sub-variety* is real analytic subset of a given regular manifold. A *regular sub-variety* is a sub-variety and a regular sub-manifold. Let \mathcal{O}_M be the sheaf of real analytic function germs on the regular manifold M .

In what follows the adjective *analytic* only means *real analytic*.

Let \underline{a} be a point of M and let $\mathcal{O}_{\underline{a}} := \mathcal{O}_{M, \underline{a}}$ be the regular local ring of the germ (M, \underline{a}) . Let $\mathfrak{m}_{\underline{a}}$ be its maximal ideal and let $n = \dim(M, \underline{a}) := \dim_{\mathbb{R}} \mathcal{O}_{\underline{a}} / \mathfrak{m}_{\underline{a}}$. A principal ideal I of $\mathcal{O}_{\underline{a}}$ is *monomial at \underline{a}* if there exists a regular sequence of parameters x_1, \dots, x_n such that I is generated by $x_1^{e_1} \cdots x_n^{e_n}$ for non-negative integers e_1, \dots, e_n .

A *normal crossing divisor of a regular manifold M of dimension n* is the co-support D of a principal \mathcal{O}_M -ideal of finite type which is locally monomial at each point of M .

Let D be a normal crossing divisor of M . At each point $\underline{a} \in M$ there exists local coordinates $(\mathbf{u}, \mathbf{v}) = (u_1, \dots, u_s; \mathbf{v})$ centered at \underline{a} , with $0 \leq s \leq n$, such that the germ of D at \underline{a} writes $(D, \underline{a}) = \{u_1 \cdots u_s = 0\}$. Such coordinates are said *adapted to the normal crossing divisor D at \underline{a}* . We will shorten as *nc-divisor*.

Let \underline{a} be a point of M . A *local monomial \mathcal{M} (at \underline{a}) in the nc-divisor (D, \underline{a})* is a function germ of $\mathcal{O}_{\underline{a}} := \mathcal{O}_{M, \underline{a}}$ such that there exists local coordinates $(\mathbf{u}, \mathbf{v}) = (u_1, \dots, u_s; \mathbf{v})$ adapted to D at \underline{a} in which $\mathcal{M} = \pm \prod_{i=1}^s u_i^{p_i}$,

for non-negative integer numbers p_1, \dots, p_s . A principal \mathcal{O}_M -ideal of finite type I is *monomial in the nc-divisor* D if at each point of M there exists local coordinates $(\mathbf{u}, \mathbf{v}) = (u_1, \dots, u_s; \mathbf{v})$ adapted to D at \underline{a} such that the local generator at \underline{a} of the ideal I is a local monomial in the nc-divisor D .

Two local monomials $\mathcal{M}_1 = \pi_{i=1}^s u_i^{p_i}$ and $\mathcal{M}_2 = \pi_{i=1}^s u_i^{q_i}$ in the nc-divisor (D, \underline{a}) are said *ordered* if, either $p_i \geq q_i$ for each i or $p_i \leq q_i$ for each i . Any finite set of local monomials in (D, \underline{a}) is *well ordered* if any pair of distinct monomials is ordered. Any finite set of monomials in the nc-divisor is well ordered if it is well ordered as a finite set of local monomials at each point of the nc-divisor.

Let D be a nc-divisor of M . A regular sub-manifold C of M is *normal crossing with D at \underline{a}* if up to a change of coordinates adapted to D at \underline{a} , we have $(C, \underline{a}) = \{u_1 = \dots = u_r = v_1 = \dots = v_t = 0\}$ for $0 \leq r \leq s$ and $0 \leq t \leq n - s$.

Let (M, E) be a regular manifold with a nc-divisor E (possibly empty). A blowing-up with center a regular sub-variety C is *geometrically admissible* if it is normal crossing with E at each point (see [4, p. 213] for a more restrictive definition). Assume the center C is of codimension greater than or equal to 2, and let $\beta_C : (M', E') \rightarrow (M, E)$ be such a blowing-up, then $E' := \beta_C^{-1}(E \cup C)$ is a nc-divisor.

Let Z be any subset of M . The strict transform of Z by β_C is defined as the analytic Zariski closure of $\beta_C^{-1}(Z \setminus C)$ and is denoted Z^{str} . If $\gamma : (M'', E'') \rightarrow (M', E')$ is a locally finite sequence of geometrically admissible blowings-up, we will again denote Z^{str} for the strict transform of Z by $\beta_C \circ \gamma$. Suppose that the nc-divisor E is the exceptional divisor of a locally finite sequence of geometrically admissible blowings-up $\pi : (M, E) \rightarrow N$, for some regular manifold N , so that $E' = E^{\text{str}} \cup \beta_C^{-1}(C)$. Nevertheless **strict transforms of an existing exceptional divisor will be denoted by the same letter as the exceptional divisor**, namely $E' = E \cup \beta_C^{-1}(C)$.

We will use, almost systematically, the following celebrated resolution of singularities of Hironaka, in the following (embedded) version in the real setting.

Theorem 2.1 (Embedded resolution of singularities [15, 1, 4]). *Let M be a (connected) regular manifold.*

1) *Let I be a (non-zero and coherent) \mathcal{O}_M -ideal sheaf. There exists a locally finite composition of geometrically admissible blowings-up $\pi : (\widetilde{M}, E_{\widetilde{M}}) \rightarrow M$ such that the total transform π^*I is a principal ideal and monomial in the nc-divisor $E_{\widetilde{M}}$.*

2) *Let X be a sub-variety of M , of codimension larger than or equal to one, for which there exists a coherent \mathcal{O}_M -ideal sheaf with co-support X . Let Y be the singular locus of X . There exists a locally finite composition of geometrically admissible blowings-up $\pi : (\widetilde{M}, \widetilde{X}, E_{\widetilde{M}}) \rightarrow (M, X, Y)$ such that $\widetilde{X} := \pi^{-1}(X \setminus Y)^{\text{str}}$ is a regular sub-variety of \widetilde{M} , normal crossing with the nc-divisor $E_{\widetilde{M}} := \pi^{-1}(Y)$ and such that $\widetilde{X} \cap E_{\widetilde{M}}$ is a nc-divisor of \widetilde{X} .*

Although real algebraic sub-varieties can always be equipped with a ringed space structure induced by a coherent ideal, in order to be desingularized real analytic singular sub-varieties (see [14, Sect. 4]) must also be equipped with a real analytic space structure, which, unlike their complex counter-parts, is not always possible. (See [4, Sect. 10] for a proper account on the category of ringed spaces that can be desingularized). Therefore we ask for the following condition to be satisfied

Hypothesis. Any singular sub-variety to be desingularized admits a coherent ideal sheaf of the structural sheaf of the ambient regular manifold with co-support the given sub-variety, so that the sub-variety is equipped with the corresponding real analytic space structure. We recall the following and very useful (and used by us) result of ordering any finite family of monomials.

Theorem 2.2 (Ordering Monomials [4]). *Let M be a (connected) regular manifold. Let D be a nc-divisor such that each of its component is regular. Let I_1, \dots, I_k be principal \mathcal{O}_M -ideals monomial in the nc-divisor D . There exists a locally finite sequence of geometrically admissible blowings-up (normal crossing with D and its iterated total transforms) $\pi : N \rightarrow M$ such that the pulled-back ideals $\pi^*I_1, \dots, \pi^*I_k$ are principal \mathcal{O}_N -ideals*

monomial in the nc-divisor $\pi^{-1}(D)$ such that at each point of N the local generators of $\pi^*I_1, \dots, \pi^*I_k$ are ordered.

Notations.

We will use *Unit* to mean any analytic function germ which is a local unit and for which a more specific notation is not necessary.

We will write *const* to mean a non-zero constant we do not want to precise further.

We will write sometimes (...) to mean a regular function germ we do not want to denote specifically.

If z is a component of some local coordinates system centered at some given point then $z^{+\infty}$ means the null function germ.

Let $\overline{\mathbb{N}} := \mathbb{N} \cup \{+\infty\}$ and $\overline{\mathbb{N}}_{\geq t} := \{n \in \overline{\mathbb{N}} : n \geq t\}$.

3. BILINEAR SYMMETRIC FORMS AND THEIR PROJECTIVE QUADRATIC FORMS

Let V be a real vector space of finite dimension.

The real vector space tensor product $V \otimes V$ decomposes as the direct sum of the real vector subspaces $\text{Sym}^2(V) \oplus \wedge^2 V$, where $\text{Sym}^2(V)$ is the 2-nd symmetric power of V and $\wedge^2 V$ is the 2-nd exterior power of V . A symmetric bilinear form of V is just an element of $\text{Sym}(2, V)$ the dual vector space of $\text{Sym}^2(V)$.

Let $Q(V)$ be the real vector space of real quadratic forms on V . For $\kappa \in \text{Sym}(2, V)$, let κ_{Δ} be the associated quadratic form. There is a canonical isomorphism $Q(V) \rightarrow \text{Sym}(2, V)$ mapping a quadratic form onto its polar form.

The vector space V is canonically equipped with a scalar product inducing a metric \mathbf{g} , we write $|\cdot|_{\mathbf{g}}$ for the norm and $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ for the scalar product. We associate to the bilinear symmetric form κ on V its *projective form*, namely the quadratic mapping $\mathbf{p}\kappa : \mathbf{P}V \rightarrow \mathbb{R}$ defined as

$$\mathbf{p}\kappa : \mathbf{P}V \ni \delta \rightarrow \mathbf{p}\kappa(\delta) := \frac{1}{|\xi|_{\mathbf{g}}^2} \kappa_{\Delta}(\xi),$$

for any $\xi \in V \setminus \mathbf{0}$ such that δ is the vector direction of the real line $\mathbb{R}\xi$.

When κ and κ' are two bilinear symmetric forms such that $\kappa' = \lambda\kappa$ with $\lambda \in \mathbb{R}^*$, then $\mathbf{p}\kappa' = \text{sign}(\lambda)\mathbf{p}\kappa$. Thus the critical loci of $\mathbf{p}\kappa$ and respectively of $\mathbf{p}\kappa'$ are the same. The space of all the maps $\mathbf{p}\kappa$ for $\kappa \in \text{Sym}(2, V)$ is $\mathbf{S}Q(V)$, the "unit sphere bundle of $Q(V)$ ", obtained in taking the quotient of $Q(V) \setminus \mathbf{0}$ by the multiplicative action of $\mathbb{R}_{>0}$.

Notations. We write $Q(\mathbf{P}V) := \mathbf{S}Q(V)$, since we would like to think of it as the space of non-zero "quadratic forms" on $\mathbf{P}V$.

Let M be a connected regular manifold of finite dimension. Let F be a regular vector bundle of finite rank over M . Let $\mathbf{P}F$ be the projective bundle associated with F . Let $\Gamma_M(F)$ be the \mathcal{O}_M -module of regular sections of F . Let $\text{Sym}^2(F)$ be the regular vector bundle over M of the 2-nd symmetric power of F . Let $\text{Sym}(2, F) := \text{Sym}^2(F)^*$, respectively $Q(F)$, be the vector bundle over M of 2-symmetric tensors, respectively quadratic forms, over the fibers of F . These two vector bundles are canonically isomorphic via a regular mapping of vector bundles over M . A regular *quadratic form on F* is a regular section $M \rightarrow Q(F)$ and a *2-symmetric regular tensor field on F* is a regular section $M \rightarrow \text{Sym}(2, F)$, shortened as *2-symmetric tensor on F* . When $F = TM$ we just say *2-symmetric tensor on M* , respectively *quadratic form on M* .

A *fiber-metric* on F is a regular section $M \rightarrow \text{Sym}(2, F)$ such that the corresponding quadratic form is everywhere positive definite. Such a fiber-metric always exists by construction. When F is equipped with a fiber metric \mathbf{g} we write $|\cdot|_{\mathbf{g}}$ for the norm and $\langle \cdot, \cdot \rangle_{\mathbf{g}}$ for the corresponding scalar product in the fiber.

Like for foliations (see Section 5), it is more convenient to study invertible sub-modules of 2-symmetric tensors than just one such section.

Let \mathcal{L} be an invertible \mathcal{O}_M -sub-module of $\Gamma_M(\text{Sym}(2, F))$. We say that \mathcal{L} *vanishes at the point* $\underline{a} \in M$ if the bilinear symmetric form $\kappa(\underline{a}) \in \text{Sym}(2, T_{\underline{a}}M)$ is the null form where κ is a local generator of \mathcal{L} nearby \underline{a} . The *vanishing locus* $V(\mathcal{L})$ of \mathcal{L} is the co-support of \mathcal{L} and is a sub-variety of M . Since M is connected, we will say that \mathcal{L} is *non-zero* (or *non-null*) if $V(\mathcal{L})$ is a proper subset of M , thus of codimension larger than or equal to one. We say that \mathcal{L} *does not vanish over a subset* S if $V(\mathcal{L}) \cap S = \emptyset$.

Let κ be a local generator of \mathcal{L} over a neighborhood \mathcal{U} of some given point. Suppose that F is equipped with the fiber-metric \mathbf{g} . Suppose that κ is non-zero (that is $\mathcal{L}|_{\mathcal{U}}$ is). We associate with κ a *projective form*, namely the mapping $\mathbf{p}\kappa : \mathcal{U} \setminus V(\mathcal{L}) \rightarrow Q(\mathbf{P}F)|_{\mathcal{U} \setminus V(\mathcal{L})}$ (see notation below) defined as

$$\mathbf{p}\kappa : \mathcal{U} \setminus V(\mathcal{L}) \ni \underline{a} \rightarrow \{\mathbf{p}\kappa(\underline{a}) := \mathbf{p}(\kappa(\underline{a})_{\Delta}) : \mathbf{P}F_{\underline{a}} \ni \delta \rightarrow (\mathbf{p}(\kappa(\underline{a})_{\Delta}))(\delta)\},$$

where $\mathbf{p}(\kappa(\underline{a})_{\Delta})$ is the projective form associated with the bilinear symmetric form $\kappa(\underline{a})$.

The mapping $\mathbf{p}\kappa$ is *constant along the fibers* if for each $\underline{a} \in \mathcal{U} \setminus V(\mathcal{L})$ the function $\mathbf{p}\kappa(\underline{a}) : \mathbf{P}F_{\underline{a}} \rightarrow \mathbb{R}$ is constant. This property depends only on \mathcal{L} so that we will say that \mathcal{L} is *constant along the fibers*.

Let \mathcal{L} be a nowhere vanishing invertible \mathcal{O}_M -module of $\Gamma_M(\text{Sym}(2, F))$. Let κ be a local generator of \mathcal{L} over some open subset \mathcal{U} of M . For $\phi \in \mathcal{O}_{\mathcal{U}}$ any local analytic unit on the open subset \mathcal{U} , we get $\mathbf{p}(\phi\kappa) = \phi\mathbf{p}\kappa$ so that for each $\underline{a} \in \mathcal{U}$ we see that $\text{crit}(\mathbf{p}(\phi\kappa)(\underline{a})) = \text{crit}(\mathbf{p}\kappa(\underline{a}))$. The *local vertical critical locus of \mathcal{L} over \mathcal{U}* , is the sub-variety (of $F|_{\mathcal{U}}$) consisting of the union over all the points \underline{a} of \mathcal{U} of the critical locus of $\mathbf{p}\kappa(\underline{a})$, for κ any local generator of \mathcal{L} namely:

$$VC(\mathcal{L}, \mathcal{U}) := \cup_{\underline{a} \in \mathcal{U}} \{\underline{a}\} \times \text{crit}(\mathbf{p}\kappa(\underline{a})) \subset Q(\mathbf{P}(F|_{\mathcal{U}})).$$

The *vertical critical locus* $VC(\mathcal{L})$ of \mathcal{L} is the union of all the local vertical critical loci $VC(\mathcal{L}, \mathcal{U})$, and it is a sub-variety of $Q(\mathbf{P}F)$.

4. GOOD PARAMETERIZATION OF 2-SYMMETRIC TENSORS ON REGULAR SURFACES

Proposition 4.5 is the main result of this section and is the first important step towards our main result. It is essentially a precise local simultaneous diagonalization result for a global orthogonal (directional) frame. What we do, though, is simpler and faster than the result of [18] due to avoiding working with matrices and being in dimension two. We moreover get the existence of the global orthogonal (directional) frame that Kurdyka & Paunescu [18] do not exhibit (but they do not look for it either).

Let S be a connected regular surface and let B a regular vector bundle of rank two over S .

We also suppose that B admits a regular fiber-metric \mathbf{g} .

Given a trivializing neighborhood $\mathcal{U} \subset S$ of B , and picking any vector bundle frame of B over \mathcal{U} , we recall, after using Gram-Schmidt ortho-normalization process for $\mathbf{g}|_{\mathcal{U}}$, that the following fact, to be kept in mind at any time up to the end of this Section, holds true:

Remark 1. *For each point \underline{a} of S , there exist a trivializing neighborhood $\mathcal{U} \simeq U \times \mathbb{R}^2$ of the vector bundle B over \underline{a} equipped with a local coordinate system $(u, v; X, Y)$ such that for each $(u, v) \in U$ we find that $(\mathbf{g}(u, v))((X_1, Y_1), (X_2, Y_2)) = X_1X_2 + Y_1Y_2$.*

Since the structural sheaf \mathcal{O}_S is coherent, any \mathcal{O}_S -ideal (sheaf) of finite type is also coherent.

Suppose given \mathcal{L} a non-zero invertible \mathcal{O}_S -module of $\text{Sym}(2, B)$. The *ideal $\mathcal{C}_{\mathcal{L}}$ of coefficients of \mathcal{L}* is the \mathcal{O}_S -ideal obtained by the evaluation of \mathcal{L} (by means of local generators) along the (local) regular sections $S \rightarrow \text{Sym}^2(B)$. The vanishing locus $V(\mathcal{L})$ is of course the co-support of $\mathcal{C}_{\mathcal{L}}$. The ideal $\mathcal{C}_{\mathcal{L}}$ is of finite type by definition (thus \mathcal{O}_S -coherent).

The *degeneracy locus* $D(\mathcal{L})$ of \mathcal{L} is the sub-variety of S consisting of the points where any local generator κ gives rise to a degenerate bilinear symmetric form (on the fiber), and contains $V(\mathcal{L})$. If $D(\mathcal{L}) = \emptyset$, we say that \mathcal{L} is *non-degenerate*. If $D(\mathcal{L}) = S$, we say that \mathcal{L} is *everywhere degenerate*. If $D(\mathcal{L})$ is everywhere of positive local codimension, we say that \mathcal{L} is *generically non-degenerate*.

Given local (and trivializing) coordinates $(u, v; X, Y)$ of B at $\mathbf{p} \in S$, we can write κ a local generator of \mathcal{L} at \mathbf{p} as

$$(\kappa(\underline{a}))((X_1, Y_1), (X_2, Y_2)) = aX_1X_2 + b(X_1Y_2 + Y_1X_2) + cY_1Y_2,$$

for function germs $a, b, c \in \mathcal{O}_{S, \mathbf{p}}$. Note that \mathcal{C}_κ is locally generated at \mathbf{p} by a, b, c . Note also that \mathcal{O}_S -ideal locally generated at (any) \mathbf{p} by $ac - b^2$ is also of finite type, whose co-support is exactly the degeneracy locus $D(\mathcal{L})$.

Notation. We denote this latter \mathcal{O}_S -ideal by $I_{\mathcal{L}}^D$.

Remark 2. Although we wrote in the introduction that we will not define what is a singular 2-symmetric tensor on B , that is a local regular section of $\text{Sym}(2, B)$ with singular-like/critical-like properties, we cannot help to point out that any such invertible module either, with the co-support of $\mathcal{C}_\mathcal{L}$ not empty and of codimension positive, or with the co-support of $I_{\mathcal{L}}^D$ not empty is definitely a singular 2-symmetric tensor.

We start with the following

Lemma 4.1. *There exists a locally finite sequence of points blowings-up $\sigma_1 : (S_1, E_1) \rightarrow S$ so that the total transform $\sigma_1^* \mathcal{C}_\mathcal{L}$ is principal and monomial in the nc-divisor $V_{\mathcal{L}} := \sigma_1^{-1}(\text{co-supp}(\mathcal{C}_\mathcal{L}))$ which contains the exceptional divisor E_1 .*

Proof. It is straightforward from principalization and monomialization of ideals [15, 3], as quoted in point 1) of Theorem 2.1. \square

Let $\sigma : R \rightarrow S$ be a locally finite composition of point blowings-up and let $E \subset R$ be the exceptional divisor. By definition, the pull-back $\sigma^* \mathcal{L}$ of \mathcal{L} by σ is the invertible \mathcal{O}_R -module of $\Gamma_R(\sigma^* \text{Sym}(2, B))$ (observe that $\sigma^* \text{Sym}(2, B) = \text{Sym}(2, \sigma^* B)$) is locally generated by $\kappa \circ \sigma$.

Since \mathcal{L} is non-zero, the total transform $\sigma^* \mathcal{C}_\mathcal{L}$ writes as $\sigma^* \mathcal{C}_\mathcal{L} = J \cdot K$ where J is principal and monomial in E while K is an ideal whose co-support does not contain any component of E . The invertible \mathcal{O}_R -submodule of $\Gamma_R(\text{Sym}(2, \sigma^* B))$ defined as $\sigma^* \mathcal{L}^{div} := J^{-1} \cdot (\sigma^* \mathcal{L})$ is called the *divided pull-back* of \mathcal{L} by σ . If κ' is a local generator of $\sigma^* \mathcal{L}^{div}$ then the form κ' is the null bilinear symmetric form at $\underline{a} \in R$ if and only if $\underline{a} \in \text{co-supp}(K)$.

As we did for \mathcal{L} we can define $D(\sigma^* \mathcal{L}^{div})$, the *degeneracy locus* of $\sigma^* \mathcal{L}^{div}$, that is the set of points \underline{a} of R where bilinear symmetric form $\kappa'(\underline{a})$ is degenerate, for κ' a local generator of $\sigma^* \mathcal{L}^{div}$ at \underline{a} . If \mathcal{L} is non-degenerate (respectively everywhere-degenerate, respectively generically degenerate), so is $\sigma^* \mathcal{L}^{div}$.

When \mathcal{L} is generically non-degenerate, we find out that $\sigma^* I_{\mathcal{L}}^D$ writes as $\sigma^* I_{\mathcal{L}}^D = J^D \cdot K^D$ where J^D is principal and monomial in E while K^D is an \mathcal{O}_R -ideal whose co-support does not contain any component of E . Since $I_{\mathcal{L}}^D \subset \mathcal{C}_\mathcal{L}^2$, the ideal J^D is contained in the ideal J^2 , and thus deduces that

$$D(\sigma^* \mathcal{L}^{div}) = \text{co-supp}(J^{-2} J^D K^D).$$

The next Lemma continues the process initiated with Lemma 4.1.

Lemma 4.2. *- If \mathcal{L} is generically non-degenerate, there exists a locally finite sequence of points blowings-up $\beta_2 : (S_2, E_2) \rightarrow (S_1, E_1)$ so that the total transform $(\sigma_1 \circ \beta_2)^* I_{\mathcal{L}}^D$ is principal and monomial in the nc-divisor $D_{\mathcal{L}} := (\sigma_1 \circ \beta_2)^{-1}(D(\mathcal{L}))$ which is normal crossing with the nc-divisor $E_2 \cup V_{\mathcal{L}}^{\text{str}}$.*

- When \mathcal{L} is not generically non-degenerate, we define $(S_2, E_2) := (S_1, E_1)$ and β_2 is the identity mapping. We additionally convene that $D_{\mathcal{L}} = \emptyset$.

Proof. It is again straightforward from principalization and monomialization of ideals. \square

Remark 3. Any component of E_2 which is not a component (of the strict transform) of E_1 is a component of $D_{\mathcal{L}}$ along which $((\sigma_1 \circ \beta_2)^* \mathcal{L})^{div}$ is non-vanishing but is degenerate.

In order to present the main result of this section, we need some preparatory material. We will work mostly locally, with germs. These local data will be gathered in an appropriate module or ideal sheaf.

Let $\sigma_2 := \sigma_1 \circ \beta_2 : (S_2, E_2) \rightarrow S$, let $B_2 := \sigma_2^* B$ and let \mathcal{L}_2 be the invertible \mathcal{O}_{S_2} -module $\sigma_2^* \mathcal{L}^{div} := (\sigma_1 \circ \beta_2)^* \mathcal{L}^{div} = \beta_2^* (\sigma_1^* \mathcal{L}^{div})$ (by Lemma 4.1) and let κ_2 be a local generator of \mathcal{L}_2 over some open neighborhood \mathcal{U} (of some point). Thus we know that κ_2 vanishes nowhere in \mathcal{U} . From Section 3, it induces the regular mapping $\mathbf{p}\kappa_2 : \mathcal{U} \rightarrow \Gamma_{S_2}(Q(\mathbf{P}B_2))|_{\mathcal{U}}$.

For each point \underline{a} , the mapping $\mathbf{p}\kappa_2(\underline{a})$ is a regular "quadratic" mapping from $\mathbf{P}(B_{\underline{a}}) = \mathbf{P}(B_{\sigma_2(\underline{a})}) = \mathbf{P}(\mathbb{R}^2)$ to \mathbb{R} . Thus if not constant it has a maximum and a minimum. Those line directions correspond to the "eigen-spaces" of the bilinear symmetric form $\kappa_2(\underline{a})$. The *vertical critical locus* $VC(\mathcal{L}_2)$ is the sub-variety of $\mathbf{P}B_2$ obtained as the union, taken over S_2 , of all these extremal line directions.

Since $\mathbf{p}\kappa_2$ is not constant along the fibers, the vertical critical locus $VC(\mathcal{L}_2)$ is a sub-variety of pure dimension 2 which project surjectively on S_2 .

Let \underline{a} be a point of S_2 . Let $\mathcal{U}_2 \simeq U_2 \times \mathbf{P}\mathbb{R}^2$ be a trivializing open neighborhood of $\mathbf{P}B_2$ such that the open neighborhood U_2 contains \underline{a} . So we can assume that we work on \mathcal{U}_2 , equipped with regular local coordinates $(u, v; [X : Y])$ inherited from those described in Remark 1 for the metric $\mathbf{g} \circ \sigma_2$. Expliciting the form $\mathbf{p}\kappa_2$ in these coordinates, a local equation of the local vertical critical locus over \mathcal{U}_2 is

$$(1) \quad VC(\mathcal{L}_2, \mathcal{U}_2) = \{(Y\partial_X - X\partial_Y)(\mathbf{p}\kappa_2(u, v)(X, Y)) = 0\} = \{c_2 X^2 + 2b_2 XY - c_2 Y^2 = 0\},$$

with $b_2, c_2 \in \mathcal{O}_{S_2, \underline{a}}$.

Let $J_2(U_2)$ be the ideal generated by b_2, c_2 . Once more all these ideals defined locally can be glued into a well defined \mathcal{O}_{S_2} -ideal $I_{\mathcal{L}_2}^{VD}$ of finite type, since in the construction above nothing is coordinate dependent. The co-support of the ideal $I_{\mathcal{L}_2}^{VD}$ is $VD(\mathcal{L}_2)$, the *vertical discriminant*. It is a sub-variety of codimension larger than or equal to one, if not empty.

Lemma 4.3. - *If \mathcal{L} is not constant along the fibers, there exists a locally finite sequence of points blowings-up $\beta_3 : (S_3, E_3) \rightarrow (S_2, E_2)$ so that the total transform $\beta_3^* I_{\mathcal{L}_2}^{VD}$ is principal and monomial in the, vertical discriminant, the nc-divisor $VD_{\mathcal{L}} := \beta_3^{-1}(\text{co-supp}(I_{\mathcal{L}_2}^{VD}))$ which is normal crossing with the nc-divisor $E_3 \cup V_{\mathcal{L}}^{\text{str}} \cup D_{\mathcal{L}}^{\text{str}}$.*

- *If \mathcal{L} is constant along the fibers then we define $S_3 := S_2$, $E_3 := E_2$ and β_3 is the identity map of S_2 .*

Proof. It is again straightforward from principalization and monomialization of ideals. \square

Let $\sigma_3 := \sigma_2 \circ \beta_3$ and $\mathcal{L}_3 := \sigma_3^* \mathcal{L}_2$ and $B_3 := \sigma_3^* B$.

Let \underline{a} be a point of S_3 . Let $\mathcal{U}_3 \simeq U_3 \times \mathbf{P}\mathbb{R}^2$ be a trivializing open neighborhood of $\mathbf{P}B_3$ such that U_3 is an open neighborhood of \underline{a} . Let $(u, v; [X : Y])$ some local coordinates on \mathcal{U}_3 coming from Remark 1 for the metric $\mathbf{g} \circ \sigma_3$. If \mathcal{L} is not constant along the fibers, a local equation of the local vertical critical locus over \mathcal{U} is

$$(2) \quad VC(\mathcal{L}_3, \mathcal{U}_3) = \{c_3 X^2 + 2b_3 XY - c_3 Y^2 = 0\},$$

with $b_3, c_3 \in \beta_3^* I_{\mathcal{L}_2}^{VD}$. Since this latter ideal is locally generated by a monomial \mathcal{M} in the nc-divisor $VD_{\mathcal{L}}$, we deduce that $b_3 = \mathcal{M} \cdot b$ and $c_3 = \mathcal{M} \cdot c$ for regular function germs b, c so that one of them is a local unit. In order to distinguish the cases c is a local unit and c is not a local unit, up to the "orthonormal" change of coordinates in $U := X + Y, V = X - Y$ (when c is not a local unit) we end up with a situation where we can always assume that c is a local unit, so that we can assume that $c \equiv 1$.

Thus we introduce the *reduced vertical critical locus* $RVC(\mathcal{L}_3)$, locally defined as

$$(3) \quad RVC(\mathcal{L}_3) = \{X^2 + 2bXY - Y^2 = 0\}$$

which consists exactly of two points $[X_1 : Y_1]$ and $[X_2 : Y_2]$ where $X_1 = Y_1(b + \sqrt{1 + b^2})$ and $X_2 = Y_2(b - \sqrt{1 + b^2})$.

Definition 4.4. A non-zero invertible sub- \mathcal{O}_S -module \mathcal{L} of $\text{Sym}(2, B)$ is said orientable if there exists a locally finite covering $(\mathcal{U}_i)_{i \in I}$ of S such that for each $i \in I$ there exists a generator κ_i of $\mathcal{L}|_{\mathcal{U}_i}$ such that for each pair $i, j \in I$ with $i \neq j$ and $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ we find $\kappa_i = \varphi_{i,j} \kappa_j$ such that the transition function $\varphi_{i,j}$ is positive on $\mathcal{U}_i \cap \mathcal{U}_j$.

The invertible \mathcal{O}_S -module generated by κ a global regular section of $\text{Sym}(2, B)$ is orientable.

We now can state the main result of this section

Proposition 4.5. Let \mathcal{L} be a non-zero and orientable invertible \mathcal{O}_S -module of $\Gamma_S(\text{Sym}(2, B))$. There exists a locally finite sequence of points blowings-up $\sigma_R : (R, E_R) \rightarrow S$ such that,

1) Assuming that \mathcal{L} is not constant along the fibers.

i) Then $\sigma_R = \sigma_3$, so that the respective conclusions of Lemma 4.1, Lemma 4.2 and Lemma 4.3 hold true.

ii) There exist two unique (up to permutation) invertible \mathcal{O}_{S_3} -submodules $\Theta_1(\mathcal{L})$ and $\Theta_2(\mathcal{L})$ of $\Gamma_{S_3}(B_3^*)$, both with empty co-support, such that for every point \underline{a} of S_3 , there exists a neighborhood \mathcal{U} of \underline{a} such that

(a) For $i = 1, 2$, let θ_i be a local regular section $\mathcal{U} \rightarrow B_3^*$ locally generating $\Theta_i(\mathcal{L})$ such that for each $\underline{b} \in \mathcal{U}$, the kernels $\ker(\theta_1(\underline{b}))$ and $\ker(\theta_2(\underline{b}))$ are both a line of $(B_3)_{\underline{b}} = B_{\sigma_3(\underline{b})}$ and are orthogonal for the metric $\mathbf{g} \circ \sigma_3$.

(b) Let κ_3 denotes a local generator of \mathcal{L}_3 over \mathcal{U} . For each $\underline{b} \in \mathcal{U}$, we find $\{[\ker(\theta_1(\underline{b}))], [\ker(\theta_2(\underline{b}))]\}$, where $[\ker \theta_i(\underline{b})]$ denote the point of $(\mathbf{P}B_3)_{\underline{b}}$ corresponding to the line $\ker \theta_i(\underline{b})$, are the critical points of $\mathbf{p}\kappa_3(\underline{b})$.

(c) The local generator κ_3 of \mathcal{L} writes over \mathcal{U} as

$$(4) \quad \kappa_3 = \varepsilon_1 \mathcal{M}_1 \theta_1 \otimes \theta_1 + \varepsilon_2 \mathcal{M}_2 \theta_2 \otimes \theta_2.$$

with

- $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$.
- If \mathcal{L}_3 is non-degenerate, then $\mathcal{M}_1 = \mathcal{M}_2$ is a monomial in $V_{\mathcal{L}}^{\text{str}} \cup E_3$ locally generating $\sigma_3^* \mathcal{C}_{\mathcal{L}}$.
- If \mathcal{L}_3 is generically non-degenerate, one of the function germ $\mathcal{M}_1, \mathcal{M}_2$ is a local monomial in $V_{\mathcal{L}}^{\text{str}} \cup E_3$ locally generating $\sigma_3^* \mathcal{C}_{\mathcal{L}}$, while the other one is a monomial in $V_{\mathcal{L}}^{\text{str}} \cup D_{\mathcal{L}}^{\text{str}} \cup E_3$ and is locally generating the $\sigma_3^* I^D(\mathcal{L})$.
- If \mathcal{L}_3 is degenerate, one of germs $\mathcal{M}_1, \mathcal{M}_2$ is a monomial in $V_{\mathcal{L}}^{\text{str}} \cup E_3$ locally generating $\sigma_3^* \mathcal{C}_{\mathcal{L}}$, and the other germs $\mathcal{M}_1, \mathcal{M}_2$ is identically zero.

2) If \mathcal{L} is constant along the fibers, given a-priori any non-zero invertible \mathcal{O}_S -sub-module Θ of $\Gamma_S(B^*)$, let \mathcal{C}_{Θ} be its ideal of coefficients,

i) The mapping σ_R (depending on the choice of Θ) factors as $\sigma_R = \sigma_3 \circ \beta_R$, through a locally finite sequence of point blowings-up $\beta_R : (R, E_R) \rightarrow (S_3, E_3)$, so that the conclusion of Lemma 4.1, Lemma 4.2, and Lemma 4.3 hold true.

ii) The ideal $\sigma_R^* \mathcal{C}_{\Theta}$ is principal and monomial in the nc-divisor $V_{\Theta} := \text{co-supp}(\sigma_R^* \mathcal{C}_{\Theta})$ which is normal crossing with $E_R \cup V_{\mathcal{L}}^{\text{str}} \cup D_{\mathcal{L}}^{\text{str}}$.

iii) Let $\mathcal{L}_R := (\sigma_R^* \mathcal{L})^{\text{div}}$ and $B_R := \sigma_R^* B$. Let Θ_1 be the \mathcal{O}_R -module of $\Gamma_R(B_R^*)$ defined as $(\sigma_R^* \mathcal{C}_{\Theta})^{-1} \sigma_R^* \Theta$ (which has empty co-support). Let Θ_2 be the sub-module of $\Gamma_R(B_R^*)$ orthogonal to Θ_1 , for the fiber-metric $\mathbf{g}^* \circ \sigma_R$ where \mathbf{g}^* is the fiber metric on B^* induced by \mathbf{g} . We get

Points (a), (b) and (c) of 1-ii) are satisfied by Θ_1, Θ_2 substituting \mathcal{L}_3 by \mathcal{L}_R , σ_3 by σ_R and $D_{\mathcal{L}}^{\text{str}}$ by V_{Θ} .

Proof. Suppose that \mathcal{L} is not constant along the fibers. We check that $\sigma_R := \sigma_3$ satisfies all the properties. We start after Lemma 4.3.

We will construct two regular orthogonal (for $\mathbf{g} \circ \sigma_3$) nowhere vanishing sections $R \rightarrow \mathbf{P}B_3$, then take their dual forms and get the desired form for the local generator of \mathcal{L}_3 .

By hypothesis the reduced vertical critical locus $C := RVC(\mathcal{L}_3)$ is a regular 2-sheeted covering over S_3 . Let $\mathcal{U}_3 \simeq U_3 \times \mathbb{R}^2$ be a local trivializing open subset of B_3 , with local (trivializing) coordinates $(u, v, [X : Y])$. Over any given point $\underline{a}_3 \in U_3$, as explained above, C consist exactly of the two points $[X_1 : Y_1]$ and $[X_2 : Y_2]$ where $X_1 = Y_1(b + \sqrt{1 + b^2})$ and $X_2 = Y_2(b - \sqrt{1 + b^2})$.

Since \mathcal{L} is not constant along the fibers, so is $\mathcal{L}_3|_{U_3}$. Thus the directions fields $\delta_1(u, v) := [X_1(u, v) : Y_1(u, v)]$ and $\delta_2(u, v) := [X_2(u, v) : Y_2(u, v)]$ are orthogonal and give rise to two regular orthogonal sections $\delta_j : U_3 \rightarrow \mathbf{P}B_3$ over U_3 , with $j = 1, 2$, which diagonalize simultaneously over U_3 any generator of the 2-symmetric bilinear form $\mathcal{L}_3|_{U_3}$ so that over U_3 we get points (a) and (b) immediately. Point (c) over U_3 consists just of tracking down which ideals has been principalized and monomialized.

Assume now we are given $(\mathcal{U}_{3,\iota})_\iota$ an atlas of B_3 such that $\mathcal{U}_{3,\iota} \simeq U_{3,\iota} \times \mathbb{R}^2$. Thus there are two local regular and orthogonal direction fields $\delta_j^\iota : U_{3,\iota} \rightarrow \mathbf{P}B_3$, with $j = 1, 2$, which diagonalize simultaneously over $U_{3,\iota}$ the form $\mathcal{L}_3|_{U_{3,\iota}}$. Thus we get two invertible-submodules \mathcal{X}_j^ι of $\Gamma_{U_{3,\iota}}(\mathbf{P}B_3|_{U_{3,\iota}})$ with $j = 1, 2$, respectively generated by δ_j^ι . Since \mathcal{L} is orientable, so is \mathcal{L}_3 (via a density argument). Thus we can glue these local invertible sub-modules into \mathcal{X}_1 and \mathcal{X}_2 two (nowhere vanishing) invertible \mathcal{O}_{S_3} -sub-modules of $\Gamma_{S_3}(\mathbf{P}B_3)$. Let $\Theta_i(\mathcal{L})$ be the dual module of \mathcal{X}_i w.r.t. the fiber-metric $\mathbf{g} \circ \sigma_3$. It is an invertible sub-module of $\Gamma_{S_3}(\mathbf{P}(B_3^*))$, and we get points (a) and (b) and (c).

Point 2) is similar to point 1) in everything once we have proved 2-ii) which is straightforward by now. The extra blowings-ups $\beta_R : (R, E_R) \rightarrow (S_3, E_3)$ are just to "make nice" the module $(\beta_R^* \Theta)^{div}$. \square

In order to conclude this section, we finish with a consequence of Proposition 4.5 echoing the results of [18] (although there Kurdyka & Paunescu first regularize the "eigen-values" and then diagonalize simultaneously).

Corollary 4.6. *Let \mathcal{L} be a non-zero invertible \mathcal{O}_S -module of $\Gamma_S(\text{Sym}(2, B))$. There exists a locally finite sequence of points blowings-up $\gamma : (S', E') \rightarrow S$ such that for each point $\underline{a} \in S'$, there exists a neighborhood \mathcal{U} of \underline{a} and two orthonormal and non-vanishing local sections $\xi_1, \xi_2 : \mathcal{U} \rightarrow \gamma^* B$ such that at each point $\underline{b} \in \mathcal{U}$, the 2-symmetric tensor κ' locally generating $\gamma^* \mathcal{L}$ is a sum of square in the basis $\xi_1(\underline{b}), \xi_2(\underline{b})$.*

Consequently when \mathcal{L} is generated by a 2-symmetric tensor κ over S , each "eigen-value" of $\kappa \circ \sigma_R$, that is the size of each generator $(\mathcal{M}_i \theta_i)^2$, is a monomial times a local unit nearby \underline{a} , with $i = 1, 2$.

5. RESOLUTION OF SINGULARITIES OF PLANE SINGULAR FOLIATIONS

Let $\mathcal{O}_2 := \mathcal{O}_{\mathbb{R}^2, \mathbf{0}}$ be the local \mathbb{R} -algebra of regular function germs at $\mathbf{0}$ the origin of \mathbb{R}^2 , and let \mathfrak{m}_2 be its maximal. Let Ω_2^1 be the \mathcal{O}_2 -module of regular differential 1-form germs at $\mathbf{0}$. Let $\nu_0(f) \in \mathbb{N} \cup \{+\infty\}$ be the multiplicity at $\mathbf{0}$ of the function germ $f \in \mathcal{O}_2$. If $f \equiv 0$, then we write $\nu_0(0) = +\infty$.

Let ξ be a germ of regular vector field at the origin $\mathbf{0}$ of \mathbb{R}^2 . Given any regular local coordinates system (x, y) centered at $\mathbf{0}$, the vector field writes as $\xi = a(x, y)\partial_x + b(x, y)\partial_y$ where $a, b \in \mathcal{O}_2$. Since we are only interested in foliations (phase portraits), up to dividing ξ by $\gcd(a, b)$, we can assume that a and b have no common factor so that any vector field of the form $Unit \cdot \xi$ gives the same foliation as ξ . The vector field ξ comes with (up to the multiplication by a regular unit) a unique dual regular differential form defining the same foliation, and defined as $\omega = bdx - ady$. Since $\gcd(a, b) = 1$, we have

$$(5) \quad \iota(a, b, \mathbf{0}) := \dim_{\mathbb{R}} \mathcal{O}_2 / (a, b) < +\infty.$$

Definition 5.1. 1) A germ of a plane foliation \mathcal{F} at the origin of \mathbb{R}^2 is the data of an invertible \mathcal{O}_2 -sub-module $\mathcal{D}_{\mathcal{F}}$ of Ω_2^1 , which is finite codimensional at the origin, that is satisfying Equation (5). For a generator $bdx - ady$ of $\mathcal{D}_{\mathcal{F}}$, let $\iota(\mathcal{F}, \mathbf{0}) := \iota(a, b, \mathbf{0})$.

2) Let S be a regular surface. A foliation \mathcal{F} on S is the data of a non-zero \mathcal{O}_S -invertible sub-module $\mathcal{D}_{\mathcal{F}}$ of Ω_S^1 such that at each point \mathbf{p} of S there exists a regular diffeomorphism germ $\phi : (S, \mathbf{p}) \rightarrow (\mathbb{R}^2, \mathbf{0})$ such that the \mathcal{O}_2 -submodule $\phi_* \mathcal{D}_{\mathcal{F}}$ of Ω_2^1 is generating a germ of plane foliation at $\mathbf{0}$.

The invertible sub-module corresponding to the germ of a foliation generated by a vector field germ ξ is just $\mathcal{O}_2\omega$, for ω its dual form. Point 1) of the Definition implies that the ξ (equivalently ω) may only vanish at $\mathbf{0}$, so that for \mathbf{p} close enough to $\mathbf{0}$ and not $\mathbf{0}$, we find $\iota(\mathcal{F}, \mathbf{p}) = 0$.

Let \mathcal{F} be a foliation on a regular surface S . The *ideal of coefficients of \mathcal{F}* is the \mathcal{O}_S -ideal $\mathcal{C}_{\mathcal{F}}$ obtained by the evaluation of any local generator of $\mathcal{D}_{\mathcal{F}}$ along the germs of regular vector field. The *singular locus of \mathcal{F}* is the sub-variety defined as $\text{sing}(\mathcal{F}) := \text{co-supp}\mathcal{C}_{\mathcal{F}}$, and is of codimension 2 if not empty. We will speak of a singular plane foliation to mean that $\text{sing}(\mathcal{F})$ is not empty. If \mathbf{p} is a singular point of \mathcal{F} , taking local analytic coordinates (x, y) at \mathbf{p} , a local generator is $b dx - a dy$ so that $\mathcal{C}_{\mathcal{F}}$ is locally generated by (a, b) , and thus condition (5) is equivalent to the existence of a positive integer l such that $\mathbf{m}_{\mathbf{p}}^l \subset \mathcal{C}_{\mathcal{F}}$.

Definition 5.2. Let \mathcal{F} be a germ of plane foliation with an isolated singularity at the origin. The origin $\mathbf{0}$ is called a *simple singularity of \mathcal{F}* , if there exist local coordinates (x, y) centered at the origin such that the local generator ω writes

$$\omega = \lambda y dx - \mu x dy + \theta,$$

with $\lambda \in \mathbb{R}, \mu \in \mathbb{R}^*, \lambda \neq \mu$ and $\mu^{-1}\lambda \notin \mathbb{Q}_{>0}$, and $\theta \in \mathbf{m}_{\mathbf{p}}^2\Omega_2^1$,

Definition 5.3. Let S be a regular surface and let D be a normal crossing divisor of S and let \mathcal{F} be a foliation on S . Let \mathbf{p} be a point of D . A local irreducible component H of the germ (D, \mathbf{p}) is called *non-di-critical* (or *invariant*) at \mathbf{p} with respect to \mathcal{F} , if H is a finite union of leaves of \mathcal{F} . Otherwise H is called *di-critical* at \mathbf{p} .

The foliation \mathcal{F} is *normal crossing with D at $\mathbf{p} \in D \setminus \text{sing}(\mathcal{F})$* if the germ (D, \mathbf{p}) is not invariant, and the union $\mathcal{L}_{\mathbf{p}} \cup D$ of the leaf $\mathcal{L}_{\mathbf{p}}$ through \mathbf{p} with D is the germ of a normal crossing divisor at \mathbf{p} .

The *singular locus of \mathcal{F} adapted to D* is defined as

$$\text{sing}(\mathcal{F}, D) := \{\mathbf{p} \in S : \mathcal{F} \text{ is not normal crossing with } D \text{ at } \mathbf{p}\}.$$

It is a closed analytic subset germ of codimension 2 (if not empty) and contains $\text{sing}(\mathcal{F})$.

Definition 5.4. A point \mathbf{p} is a *simple singularity of \mathcal{F} adapted to D* (*adapted singularity* when \mathcal{F} and D are clearly identified) if, it is a simple singularity for \mathcal{F} , it belongs to D and each irreducible component of D at \mathbf{p} is non-di-critical.

Early warning on notations. In this section and in the following one, we recall that when θ is a differential 1-form on a manifold N (or a sub-module of Ω_N^1), the notation $\sigma^*\theta$ means the pull back by a given regular mapping $\sigma : M \rightarrow N$ in the sense of differential topology, that is $\sigma^*\theta := \theta \circ D\sigma$ where $D\sigma$ is the differential mapping of σ . This notation will change in Section 9 and Section 10 to avoid confusion with the pull-back in the sense of modules.

Let \mathcal{F} be a singular foliation at the origin $\mathbf{0}$ of \mathbb{R}^2 , given by the local generator $\omega = b dx - a dy$. We would like to point-out the behavior of the intersection number $\iota(\mathcal{F}, \mathbf{0})$ under point blowings-up.

Let $\pi : S_0 := [\mathbb{R}^2, \mathbf{0}] \rightarrow \mathbb{R}^2$ be the blowing-up of the origin $\mathbf{0}$ and let E_0 be the exceptional curve $\pi^{-1}(\mathbf{0})$. Let I_{E_0} be the \mathcal{O}_{S_0} -ideal of the regular function germs vanishing on E_0 . There exists a positive integer η such that $I_{E_0}^{-\eta}\pi^*\mathcal{D}_{\mathcal{F}}$ is a \mathcal{O}_{S_0} -invertible sub-module of $\Omega_{S_0}^1$, which is finite co-dimensional everywhere. The following classic Lemma tells us that this sub-module, indeed, defines a foliation on S_0 , denoted $\pi^*\mathcal{F}$. It is also fundamental tool in the resolution of singularities of plane foliations and it will also be of key importance in the proofs of our main result (although hidden) in Section 9:

Lemma 5.5 (Noether's Lemma). Let \mathcal{F} be a germ of plane foliation singular at $\mathbf{0}$.

(n-d) If E_0 is non-di-critical for $\pi^*\mathcal{F}$, then

$$\iota(\mathcal{F}, \mathbf{0}) = \nu^2 - (\nu + 1) + \sum_{\mathbf{p} \in \pi^{-1}(\mathbf{0})} \iota(\pi^*\mathcal{F}, \mathbf{p});$$

(d) If E_0 is di-critical for $\pi^*\mathcal{F}$, then

$$\iota(\mathcal{F}, \mathbf{0}) = (\nu + 1)^2 - (\nu + 2) + \sum_{\mathbf{p} \in \pi^{-1}(0)} \iota(\pi^* \mathcal{F}, \mathbf{p});$$

Remark 4. Let $\beta : (S, E) \rightarrow (\mathbb{R}^2, \mathbf{0})$ be a finite composition of point blowings-up, where S is a regular surface and the exceptional divisor $E := \beta^{-1}(\mathbf{0})$ is a nc-divisor. Noether's Lemma 5.5 implies that the ideal $\mathcal{C}_{\beta^* \mathcal{D}_{\mathcal{F}}}$ of coefficients of the \mathcal{O}_S -sub-module $\beta^* \mathcal{D}_{\mathcal{F}}$ decomposes into a product of \mathcal{O}_S -ideals $J \cdot K$, where the ideal J is principal and monomial in the exceptional divisor E , while the ideal K (with co-support in E) is a finite co-dimensional, namely $\dim_{\mathbb{R}} \mathcal{O}_{S, \mathbf{p}} / K < +\infty$ for any $\mathbf{p} \in S$.

Remark 4 leads to the following

Definition 5.6. Let $\beta : (S, E) \rightarrow (\mathbb{R}^2, \mathbf{0})$ be a finite composition of point blowings-up.

The pulled-back foliation $\beta^* \mathcal{F}$ of \mathcal{F} , is given by the invertible \mathcal{O}_S -sub-module $\mathcal{D}_{\beta^* \mathcal{F}} := J^{-1} \beta^* \mathcal{D}_{\mathcal{F}}$ of Ω_S^1 . For a local generator ω of \mathcal{F} at $\mathbf{0}$, there exists a monomial \mathcal{M} in the exceptional divisor E (generating J) and a local generator θ of $\mathcal{D}_{\beta^* \mathcal{F}}$ such that $\mathcal{M}\theta = \beta^* \omega$.

The local generator $\mathcal{M}^{-1} \beta^* \omega$ is called the strict transform of ω under the blowing-up β .

We can now present the theorem of reduction of singularities of singular plane foliation ([22, 8, 5, 7]) in the form which is most convenient for our later use. We just present the local version since the global version is easily deduced from the local one.

Theorem 5.7 ([22, 8, 5, 7]). Let \mathcal{F} be a germ of singular plane foliation at the origin $\mathbf{0}$ of \mathbb{R}^2 . There exists a finite composition of points blowings-up $\pi : (S', E') \rightarrow (\mathbb{R}^2, \mathbf{0})$ such that each point of $\text{sing}(\mathcal{F}', E')$ of the lifted foliation $\mathcal{F}' := \pi^* \mathcal{F}$ is a simple singularity of \mathcal{F}' adapted to E' .

Moreover if β is the point blowing-up $\beta : (S'', E'') \rightarrow (S', E' \cup \{\mathbf{p}'\})$ with center \mathbf{p}' , then $\mathcal{F}'' := \beta^* \mathcal{F}'$ only admits simple singularities adapted to E'' .

We end this section with the normal form of a local generator of a "desingularized" germ of singular plane foliation \mathcal{F} adapted to an exceptional divisor E as in Theorem 5.7. Observe that the singular locus $\text{sing}(\mathcal{F})$ is contained in the exceptional divisor E , by Remark 4.

- Let $\mathbf{p} \notin E$, then $\omega(\mathbf{p}) \neq 0$.
- Suppose $\mathbf{p} \in E \setminus \text{sing}(\mathcal{F})$. There exists local coordinates (u, v) centered at \mathbf{p} such that $\{u = 0\} \subset (E, \mathbf{p}) \subset \{uv = 0\}$ and $\omega(\mathbf{p}) \neq 0$.

If $(E, \mathbf{p}) = \{u = 0\}$ is invariant for \mathcal{F} , a local generator is of the form $\omega = du + u(\dots)dv$.

If $(E, \mathbf{p}) = \{u = 0\}$ is normal crossing with \mathcal{F} , a local generator is of the form $\omega = dv$ (up to a change of coordinate of the form $\bar{v} = v + F$, with $F \in \mathfrak{m}_{\mathbf{p}}$).

If $(E, \mathbf{p}) = \{uv = 0\}$. If a local generator is of the form $du + \text{Unit} \cdot dv$, we check that blowing-up \mathbf{p} will give a local generator of the pulled-backed foliation such that at each of the new two corners one of the new exceptional divisor and the strict transform of the corresponding old component through \mathbf{p} , is invariant and the other one is di-critical. Thus (up to blowing-up the point \mathbf{p}), we deduce that, up to permuting u and v , a local generator is given by $du + u(\dots)dv$ (see Lemma 6.2 and Lemma 6.3 for details).

- Suppose $\mathbf{p} \in E \cap \text{sing}(\mathcal{F})$. Thus each component of E must be invariant. There exist local coordinates (u, v) centered at \mathbf{p} such that $\{u = 0\} \subset (E, \mathbf{p}) \subset \{uv = 0\}$ and there exists another set of local coordinates (x, y) centered at \mathbf{p} such that $\omega = \lambda x dy - y dx + \theta$ where $\theta \in \mathfrak{m}_{\mathbf{p}}^2 \Omega_{\mathbf{p}}^1$ and $\lambda \notin \mathbb{Q}_{>0}$.

If $(E, \mathbf{p}) = \{uv = 0\}$, since it is non-di-critical, we deduce also that, up to permuting u and v , we get $\omega = v du + u(\dots)dv$.

If $(E, \mathbf{p}) = \{u = 0\}$, then a local generator writes as $\omega = u A dv + (u B + v^k \phi(v)) du$, with $A, B \in \mathcal{O}_{\mathbf{p}}$, where ϕ is an analytic unit in a single variable and $k = 1$ if $A(\mathbf{p}) = 0$.

When $A(\mathbf{p}) \neq 0$, a local generator is of the form

$$(6) \quad u dv + (v^k \phi(v) + u B) du$$

and when $A(\mathbf{p}) = 0$, it is of the form

$$(7) \quad (v + b_0 u)du + u\theta$$

where $\theta \in \mathfrak{m}_{\mathbf{p}}\Omega_{\mathbf{p}}^1$ and $B \in \mathcal{O}_{\mathbf{p}}$ and $b_0 \in \mathbb{R}$.

6. PAIRS OF SINGULAR FOLIATIONS AND SINGULAR FOLIATION ADAPTED TO NC-DIVISORS

The material presented in this section may or may not be new. It is hard to believe that none of the technical results stated and proved below (although tailored for our purpose) are not, even partially, already folklore of desingularization of singular plane foliations. They are simple, reasonable to expect and useful.

Let S be a regular surface. Let \mathcal{F} be a singular foliation on S .

From Section 5, any singular foliation can be resolved into a singular foliation with singularities adapted to the exceptional divisor of the resolution mapping. Moreover adapted singularities transform under blowings-up either in adapted singularities or in non-singular points.

An additional property of the resolution of a singular foliation we can wish for is a good behavior of the foliation with respect to (the pull-back of an à-priori) given curve. Thus we introduce the following useful

Definition 6.1. *Let D be a nc-divisor. A plane foliation \mathcal{F} is adapted to the nc-divisor D if:*

- *The germ (D, \mathbf{p}) at a regular point \mathbf{p} of D is either normal crossing with \mathcal{F} or is a singular point of \mathcal{F} adapted to (D, \mathbf{p}) (and thus (D, \mathbf{p}) is invariant w.r.t. \mathcal{F}).*
- *A corner point \mathbf{p} of D is either a singular point of \mathcal{F} adapted to D or it is a regular point of \mathcal{F} such that one local component of (D, \mathbf{p}) is invariant w.r.t. \mathcal{F} and the other one is normal crossing with \mathcal{F} at \mathbf{p} .*

Observe that once $\sigma : (S', E') \rightarrow S$ is a resolution of singularities of \mathcal{F} such that each singular point of $\sigma^*\mathcal{F}$ is adapted to E' , following Definition 6.1, the pulled-back foliation $\sigma^*\mathcal{F}$ is adapted to E' . A consequence of this definition is the next

Lemma 6.2. *Let D be a nc-divisor of S and let \mathcal{F} be a foliation on S adapted to D . Let $\beta : (S', E') \rightarrow S$ be the blowing-up with center a given point \mathbf{p} of S . The pulled-back foliation $\beta^*\mathcal{F}$ is adapted to the nc-divisor $\beta^{-1}(D)$.*

Proof. If \mathbf{p} does not belong to D , it is true.

Let (u, v) be local coordinates centered at the corner point $\mathbf{p} \in D$ and adapted to D , so that the germ (D, \mathbf{p}) writes $\{uv = 0\}$. We use the normal form at the end of Section 5.

If \mathbf{p} is a singular point of \mathcal{F} adapted to D , then, up to permuting u and v , a local generator of \mathcal{F} at \mathbf{p} is $\theta = vdu + uAdv$ with $A \in \mathcal{O}_{\mathbf{p}}$ such that $-A(\mathbf{p}) \notin \mathbb{Q}_{>0}$.

In the chart $(x, y) \rightarrow (x, xy)$ of the blowing-up β , we get $\beta^{-1}(D) = \{xy = 0\}$ and $\beta^*\theta = x \cdot \text{Unit} \cdot (ydx + xBdy)$, so that in this chart $\beta^*\mathcal{F}$ is adapted to $\beta^{-1}(D)$.

In the chart $(x, y) \rightarrow (xy, y)$, a similar computation yields that $\beta^*\mathcal{F}$ is also adapted to $\beta^{-1}(D)$.

If \mathbf{p} is a regular point of \mathcal{F} then, up to permuting u and v , a local generator of \mathcal{F} at \mathbf{p} is $\theta = du + uAdv$ with $A \in \mathcal{O}_{\mathbf{p}}$ such that $-A(\mathbf{p}) \notin \mathbb{Q}_{>0}$.

In the chart $(x, y) \rightarrow (x, xy)$ of the blowing-up β , we get $\beta^{-1}(D) = \{xy = 0\}$ and $\beta^*\theta = \text{Unit} \cdot (dx + xBdy)$, so that in this chart $\beta^*\mathcal{F}$ is adapted to $\beta^{-1}(D)$.

In the chart $(x, y) \rightarrow (xy, y)$ of the blowing-up β , we get $\beta^{-1}(D) = \{xy = 0\}$ and $\beta^*\theta = \text{Unit} \cdot (ydx + xBdy)$ with $B(0, 0) = 1$, so that in this chart $\beta^*\mathcal{F}$ is also adapted to $\beta^{-1}(D)$.

When \mathbf{p} is regular point of D , similar computations will lead to the stated conclusion, using once more the normal form at the end of Section 5. □

Let D be a nc-divisor of S and let \mathcal{F} be a foliation on S . Let $NA(\mathcal{F}, D)$ be the subset of points of S where the foliation \mathcal{F} is not adapted to the germ of D at this point. It is a real analytic set which is isolated, when not empty.

Lemma 6.3. *There exists a locally finite sequence of points blowings-up $\sigma_1 : (S_1, E_1) \rightarrow S$ such that the pulled-back foliation $\sigma_1^* \mathcal{F}$ is adapted to the nc-divisor $E_1 \cup D^{\text{str}}$.*

Proof. First, we resolve the singularities of \mathcal{F} , if any, by a locally finite sequence of blowings-up $\beta : (S', E') \rightarrow S$, so that \mathcal{F}' the pulled-back foliation $\beta^* \mathcal{F}$ is adapted to E' . Thus $D' := D^{\text{str}}$ is a nc-divisor which is normal crossing with E' (up to further points blowings-up). Note that $D' \cap E'$ consists only in isolated points.

Since $NA(\mathcal{F}', D')$ is isolated we can suppose that $NA(\mathcal{F}', D')$ is reduced to the single point $\{\mathbf{p}'\}$. Let (u', v') be local coordinates centered at \mathbf{p}' and adapted to D' so that $\{v' = 0\} \subset (D', \mathbf{p}') \subset \{u'v' = 0\}$.

1) Let \mathbf{p}' be a point of $D' \cap E'$, so that it is a regular point of D' . Suppose that u' is such that $(E', \mathbf{p}') := \{u' = 0\}$.

a) Suppose that \mathbf{p}' is a regular point of \mathcal{F}' . Thus the leaf through \mathbf{p}' is tangent to D' (thus normal crossing with E') while all the nearby ones are normal crossing with D' . A local generator of \mathcal{F}' is of the form

$$Unit \cdot [(u')^l + v'(\dots)]du' + dv'$$

for l a positive integer.

If we blow-up the point \mathbf{p}' , we see that the exceptional curve C'' is a maximal invariant curve of the strict transform \mathcal{F}'' of \mathcal{F}' and is normal crossing with $D'' := (D')^{\text{str}}$. In the chart $(u'', v'') \rightarrow (u'', u''v'')$ we find $D'' = \{v'' = 0\}$ with $C'' = \{u'' = 0\}$, so that a local generator at $\mathbf{p}'' = (0, 0)$ of \mathcal{F}'' is of the form

$$Unit \cdot [(u'')^{l-1} + v''(\dots)]du'' + dv''.$$

The strict transform D'' does not meet with the domain of the other blowing-up chart. We are in the same situation with $\mathbf{p}'', C'' \cup E', D''$ as we were with \mathbf{p}', E', D' , but with the exception that the exponent l in the local generator of \mathcal{F}'' at \mathbf{p}'' has dropped by 1. With further $l - 1$ point blowings-up (each center being the intersection of the latest strict transform of D' with the latest created exceptional curve) we see in the (only interesting) chart $(\mathbf{u}, \mathbf{v}) \rightarrow (\mathbf{u}, \mathbf{u}^{l-1}\mathbf{v}) = (u'', v'')$. We check that a local generator of the pull-back of \mathcal{F}'' at the terminal point of this process $\mathbf{p}^l = (0, 0)$ is of the form $d\mathbf{u} + \omega$, where $\omega \in \mathfrak{m}_{\mathbf{p}^l} \mathcal{O}_{\mathbf{p}^l}$. Thus there exists $\beta_1 : (S'_1, E'_1) \rightarrow (S', E')$ factorizing through the blowing-up of the point \mathbf{p}' so that $E'_1 = E' \cup \beta_1^{-1}(\mathbf{p}')$ so that there exists a neighborhood \mathcal{U}'_1 of the exceptional divisor $\beta_1^{-1}(\mathbf{p}')$ such that the restricted pulled-back foliation $(\beta_1^* \mathcal{F}')|_{\mathcal{U}'_1}$ is adapted to $(D^{\text{str}} \cup E'_1) \cap \mathcal{U}'_1$ (note that D^{str} is also $(D')^{\text{str}}$ and is $(D'')^{\text{str}}$ as well).

b) Suppose now that \mathbf{p}' is a singular point of \mathcal{F}' . The curve (E', \mathbf{p}') is invariant while, any leaf of \mathcal{F}' through any point $\mathbf{q} \in D'$ nearby \mathbf{p}' is normal crossing with D' . Under the current hypotheses, a local generator θ of \mathcal{F}' is of the form

$$[Unit \cdot (v')^k + (u')^l + u'v'(\dots)]du' + u'A'dv' \text{ with } k, l \in \mathbb{N}_{\geq 1} \text{ and } A' \in \mathcal{O}_{\mathbf{p}'}.$$

We blow-up the point \mathbf{p}' . In the chart $(u'', v'') \rightarrow (u''v'', v'')$, the strict transform of D' is empty. In the chart $(u'', v'') \rightarrow (u'', u''v'')$, the strict transform of D' is $D'' := \{v'' = 0\}$ and the exceptional divisor is just $C'' := \{u'' = 0\}$. Let $\mathbf{p}'' = (0, 0)$. If \mathcal{F}'' is the pull-back of \mathcal{F}' , then we see that a local generator of \mathcal{F}'' is of the form

$$[Unit \cdot (v'')^k + (u'')^{l-1} + u''v''(\dots)]du'' + u''A''dv'' \text{ where } A'' \in \mathcal{O}_{\mathbf{p}''}.$$

we conclude as in case 1) with further $l - 1$ points blowings-up.

2) Suppose that \mathbf{p}' lies in $D' \setminus E'$. Thus it is a regular point of \mathcal{F}' .

A simple computation, distinguishing between a regular point or a corner point of D' , shows that after blowing-up the point \mathbf{p}' we are in the situation 1). \square

Suppose we are given two (possibly singular) foliations \mathcal{F}_1 and \mathcal{F}_2 on the regular surface S . Let Θ_i be the invertible sub-module of Ω_S^1 corresponding to \mathcal{F}_i , for $i = 1, 2$. Suppose that these two foliations are transverse to each other on an analytic Zariski dense open set of S . Let $\Theta_{1,2}$ be the invertible \mathcal{O}_S -module of Ω_S^2 generated by $\Theta_1 \wedge \Theta_2$. Let $\mathcal{C}_{1,2}$ be the ideal of its coefficients, so that $\Theta_{1,2} = \mathcal{C}_{1,2}\Omega_S^2$. Let $\Sigma(\mathcal{F}_1, \mathcal{F}_2)$ be its co-support. Thus we find the following expected

Proposition 6.4. *Let C be a connected real analytic curve in S which is neither contained in $\Sigma(\mathcal{F}_1, \mathcal{F}_2)$ nor contains any component of $\Sigma(\mathcal{F}_1, \mathcal{F}_2)$. There exists $\tau : (S', E') \rightarrow (S, E)$ a locally finite sequence of point blowings-up such that*

- i) *The strict transform C^{str} of C is a nc-divisor which is normal crossing with E' .*
- ii) *The ideal $\tau^*\mathcal{C}_{1,2}$ is principal and monomial in its co-support the nc-divisor $\tau^{-1}(\Sigma(\mathcal{F}_1, \mathcal{F}_2))$ and is normal crossing with $E' \cup C^{\text{str}}$.*
- iii) *Let \mathcal{F}'_i be the pull-backed foliation of \mathcal{F}_i , for $i = 1, 2$. The singularities of \mathcal{F}'_i are adapted to the nc-divisor $C^{\text{str}} \cup E' \cup \tau^{-1}(\Sigma(\mathcal{F}_1, \mathcal{F}_2))$, and each component of $C^{\text{str}} \cup E' \cup \tau^{-1}(\Sigma(\mathcal{F}_1, \mathcal{F}_2))$ is either invariant or di-critical. We can further demand that each foliation $\mathcal{F}'_1, \mathcal{F}'_2$ is adapted to the nc-divisor $C^{\text{str}} \cup E' \cup \tau^{-1}(\Sigma(\mathcal{F}_1, \mathcal{F}_2))$.*
- iv) *Let Θ'_i be the $\mathcal{O}_{S'}$ -sub-module of $\Omega_{S'}^1$ associated with \mathcal{F}'_i . Thus $\Theta'_1 \wedge \Theta'_2 = \mathcal{J}'_{1,2}\Omega_{S'}^2$, with $\mathcal{J}'_{1,2}$ a principal ideal and monomial in the nc-divisor $\text{co-supp}(\mathcal{J}'_{1,2})$ (contained in the nc-divisor $\subset (\tau^{-1}(\Sigma(\mathcal{F}_1, \mathcal{F}_2)))^{\text{str}}$), and contains the ideal $\tau^*\mathcal{C}_{1,2}$.*

Proof. First we resolve the singularities of C by a locally finite sequence of points blowings-up $\sigma_1 : (S_1, E_1) \rightarrow S$.

Second we principalize and monomialize the ideal $\sigma_1^*\mathcal{C}_{1,2}$ by means of a locally finite sequence of point blowings-up $\beta_2 : (S_2, E_2) \rightarrow (S_1, E_1)$ so that $\text{co-supp}(\sigma_2^*\mathcal{C}_{1,2})$ is a nc-divisor which is normal crossing with $E_2 \cup C^{\text{str}}$ and where $\sigma_2 := \sigma_1 \circ \beta_2$.

Third, by means of a locally finite sequence of points blowings-up $\beta_3 : (S_3, E_3 = E_2 \cup E'_3) \rightarrow (S_2, E_2)$ where E'_3 is the new exceptional divisor (and keeping denoting E_2 for the strict transform E_2^{str}), the pulled-back foliation $\mathcal{F}'_i := \beta_3^*(\sigma_2^*\mathcal{F}_i)$ only have singularities adapted to E'_3 where $i = 1, 2$ (and is regular at each point of $E_2 \setminus E'_3$).

We do further point blowings-up $\beta_4 : (S_4, E_4) \rightarrow (S_3, E_3)$ so that the foliations $\beta_4^*\mathcal{F}'_i$ are adapted to the nc-divisor $E_4 \cup C^{\text{str}} \cup \Sigma^{\text{str}}$, which is possible thanks to Lemma 6.2 and Lemma 6.3. Then we define $\mathcal{F}'_i := \beta_4^*\mathcal{F}'_i$ and $\tau := \sigma_2 \circ \beta_3 \circ \beta_4$.

Point iv) is just a consequence (as a part) of the proof of point ii) that is worth putting forward. \square

The next proposition is of interest for our main result. But we need some preparatory material.

Let E be a normal crossing divisor of S .

Definition 6.5 (see [21]). *Let $\underline{a} \in S$ and let $\{h = 0\}$ be a local reduced equation of (E, \underline{a}) . A meromorphic differential q -form ω is logarithmic (along (E, \underline{a})) if $h\omega$ and $h d\omega$ are both regular q -forms.*

We denote $\Omega_S^q(\log E)$ the \mathcal{O}_S -module of q -logarithmic forms along E .

If $\mathbf{p} \notin E$, there exists a neighborhood \mathcal{U} of \mathbf{p} such that $\Omega_{\mathcal{U}}^1 = \Omega_S^1|_{\mathcal{U}} = \Omega_S^1(\log E)|_{\mathcal{U}}$.

If $\mathbf{p} \in E$ such that we can find local coordinates (u, v) at \mathbf{p} adapted to E such that $(E, \mathbf{p}) = \{u = 0\}$, there exists a neighborhood \mathcal{U} of \mathbf{p} such that $\Omega_S^1(\log E)|_{\mathcal{U}} = \mathcal{O}_{\mathcal{U}}dv + \mathcal{O}_{\mathcal{U}}d_{\log}u$ where

$$d_{\log}u := \frac{du}{u}.$$

If $\mathbf{p} \in E$, we can find local coordinates (u, v) at \mathbf{p} adapted to E such that $(E, \mathbf{p}) = \{uv = 0\}$, there exists a neighborhood \mathcal{U} of \mathbf{p} such that $\Omega_S^1(\log E)|_{\mathcal{U}} = \mathcal{O}_{\mathcal{U}}d_{\log}u + \mathcal{O}_{\mathcal{U}}d_{\log}v$.

In particular Ω_S^1 is a sub-module of $\Omega_S^1(\log E)$. If Θ is any sub-module of Ω_S^1 , it is a sub-module of $\Omega_S^1(\log E)$. A local logarithmic generator of Θ is a local generator of Θ as a sub-module of $\Omega_S^1(\log E)$.

The *ideal of logarithmic coefficients* of Θ is the ideal $\mathcal{C}_\Theta^{\log}$ locally generated by the logarithmic generator of Θ (that is the generators are written in the logarithmic basis of $\Omega_S^1(\log E)$) evaluated along local regular vector fields on S . Note that if \mathcal{C}_Θ is the ideal of coefficients of Θ , then $\mathcal{C}_\Theta^{\log} \subset \mathcal{C}_\Theta$.

Assume Proposition 6.4 is satisfied.

Let \mathbf{p}' be a corner point of E' . Let (u, v) be local coordinates centered at \mathbf{p}' and adapted to E' . Thus for each i , the foliation \mathcal{F}'_i has a local generator at \mathbf{p}' either of the form $du + u(\cdots)dv$ (up to permuting u and v) or \mathbf{p}' is an adapted singularity of \mathcal{F}'_i . Let θ_i be a local generator of Θ'_i . The *logarithmic generator* θ_i^{\log} of Θ'_i associated to θ_i is defined as follows: If $\theta_i = du + u(\cdots)dv$ then $\theta_i^{\log} := u^{-1}\theta_i = d_{\log}u + v(\cdots)d_{\log}v$ and if $\theta_i = vdu + u(\cdots)dv$ then $\theta_i^{\log} := u^{-1}v^{-1}\theta_i = d_{\log}u + (\cdots)d_{\log}v$. Note that, in each case, the logarithmic 1-form θ_i^{\log} is nowhere vanishing.

Let \mathcal{M}_i be a local monomial (in E') generating the ideal $\tau^*\mathcal{C}_{\Theta_i}$ and let \mathcal{M}_i^{\log} be a local generator of $\mathcal{C}_{\tau^*\Theta_i}^{\log}$, the logarithmic coefficient ideal of the total transform $\tau^*\Theta_i$. According to the two cases to distinguish, we either find that $\mathcal{M}_i^{\log} = u \cdot \mathcal{M}_i$ or, respectively, that $\mathcal{M}_i^{\log} = uv \cdot \mathcal{M}_i$.

Proposition 6.6. *Continuing Proposition 6.4, let \mathbf{p}' be a corner point of the exceptional divisor E' .*

v) *If \mathbf{p}' is an adapted singular point of \mathcal{F}'_1 and \mathcal{F}'_2 such that the ideals $\mathcal{C}_{\tau^*\Theta_1}$ and $\mathcal{C}_{\tau^*\Theta_2}$ are not ordered at \mathbf{p}' , there exists a locally finite sequence of point blowings-up $\beta'' : (S'', E'' = E' \cup E_{\beta''}) \rightarrow (S', E')$ such that at each corner point of $E_{\beta''}$ the ideals $\mathcal{C}_{(\tau \circ \beta'')^*\Theta_1}$ and $\mathcal{C}_{(\tau \circ \beta'')^*\Theta_2}$ are ordered. Consequently so are the ideals of logarithmic coefficients $\mathcal{C}_{(\tau \circ \beta'')^*\Theta_1}^{\log}$ and $\mathcal{C}_{(\tau \circ \beta'')^*\Theta_2}^{\log}$.*

vi) *If \mathbf{p}' is a regular point of \mathcal{F}'_1 such that the ideals $\mathcal{C}_{\tau^*\Theta_1}$ and $\mathcal{C}_{\tau^*\Theta_2}$ are not ordered at \mathbf{p}' , there exists a locally finite sequence of point blowings-up $\beta'' : (S'', E'' = E' \cup E_{\beta''}) \rightarrow (S', E')$ such that at each corner point of $E_{\beta''}$ the ideals $\mathcal{C}_{(\tau \circ \beta'')^*\Theta_1}$ and $\mathcal{C}_{(\tau \circ \beta'')^*\Theta_2}$ are ordered. Thus the ideals $\mathcal{C}_{(\tau \circ \beta'')^*\Theta_1}^{\log}$ and $\mathcal{C}_{(\tau \circ \beta'')^*\Theta_2}^{\log}$ are also ordered.*

Proof. We recall that $\mathcal{J}_i := \mathcal{C}_{\tau^*\Theta_i}$, so that $\mathcal{C}_{\tau^*\Theta_i}^{\log} = (uv) \cdot \mathcal{J}_i$.

Suppose \mathbf{p}' is such that it is an adapted singularity of both foliations.

Let (u, v) be local coordinates adapted to E' , thus $(E', \mathbf{p}') = \{uv = 0\}$. Thus \mathcal{J}_i is locally generated by $u^{p_i}v^{q_i}$ for non-negative integers p_i, q_i and $i = 1, 2$. Suppose that $p_1 < p_2$ and $q_2 < q_1$.

Let $\theta \in \Omega_{\mathbf{p}'}^1$ such that \mathbf{p}' is a singularity adapted to (E', \mathbf{p}') . Thus we can assume (up to permuting u and v) that $\theta = vdu - \lambda u dv + uv\eta$ where $\eta \in \Omega_{\mathbf{p}'}^1$ and $\lambda \notin \mathbb{Q}_{>0}$. Let γ be the blowing-up of \mathbf{p}' . At any (of the two) corner point \mathbf{p}'' of $E'' := E' \cup \gamma^{-1}(\mathbf{p}')$, let z be a reduced equation of $\gamma^{-1}(\mathbf{p}')$. We find that $\sigma^*\theta = z \cdot \theta'$ for $\theta' \in \Omega_{\mathbf{p}''}^1$ and (following Lemma 6.2) such that \mathbf{p}'' is an adapted singularity of θ' .

Let ω_i be a local generator of \mathcal{F}_i . Let \mathbf{p}'' be a corner point of E'' . Let (t, z) be local coordinates at \mathbf{p}'' adapted to E'' such that $(\gamma^{-1}(\mathbf{p}'), \mathbf{p}'') = \{z = 0\}$. Thus we deduce that $(\tau \circ \gamma)^*\omega_i = t^{r_i}z^{r_i+s_i+1}\omega'_i$ where $(r_i, s_i) = (p_i, q_i)$ or (q_i, p_i) and for a local generator ω'_i of $\gamma^*\mathcal{F}'_i$ which has an adapted singularity at \mathbf{p}'' . Thus either $(r_1 + s_1 + 1) - (r_2 + s_2 + 1)$ and $r_1 - r_2$ have the same sign or $|(r_1 + s_1 + 1) - (r_2 + s_2 + 1)| < |r_1 - r_2|$.

Thus with further finitely many point blowings-up, we find that at any corner point \mathbf{q} of the exceptional divisor lying over \mathbf{p}' , the coefficients ideals of the total transform of Θ_1 and Θ_2 are ordered.

Let \mathbf{p}' be a corner point of E' such that it is an adapted singularity of only one of the two foliations, say \mathcal{F}_2 . Let (u, v) be local coordinates adapted to E' , thus $(E', \mathbf{p}') = \{uv = 0\}$ and such that a local generator θ_1 of \mathcal{F}'_1 is of the form $du + uCdv$, so that $G := \{u = 0\}$ is invariant for \mathcal{F}'_1 and $H := \{v = 0\}$ is di-critical. Let just write a local generator θ_2 of \mathcal{F}'_2 as $vAdu + uBdv$ and (at least) one of the function germ A, B is a local analytic unit.

Suppose that for $i = 1, 2$ the ideal \mathcal{J}_i is locally generated by $u^{p_i}v^{q_i}$ for non-negative integers p_i, q_i and $i = 1, 2$ with $(p_1 - p_2)(q_1 - q_2) < 0$.

Let γ be the blowing-up with center \mathbf{p}' . Note that for $i = 1, 2$ the logarithmic generator θ_i^{\log} is of the form $a_i d_{\log}u + b_i d_{\log}v$, and at least one of the function germs a_i, b_i is a local analytic unit.

We observe that for $i = 1, 2$ that the pull-back $\gamma^* \theta_i^{\log}$ of θ_i^{\log} is of the form $a'_i d_{\log} u + b'_i d_{\log} v$ and again one of the function germs a'_i, b'_i is a local analytic unit. This means that the ideal of logarithmic coefficients $(\tau \circ \gamma)^* \Theta_i$ is just the pull-back $\gamma^*(\mathcal{C}_{(\tau \circ \gamma)^* \Theta_i})$ of the ideal of logarithmic coefficients of $\tau^* \Theta_i$. We have thus replaced a problem of foliations into a problem on (principal and monomial) ideals to order, which can be achieved by finitely many point blowings-up.

If \mathbf{p}' is a corner point of E' such that it is a regular point of both foliations \mathcal{F}'_1 and \mathcal{F}'_2 , then we reach the conclusion as in the case of a single regular foliation. \square

We end-up the section with some pairs of normal forms at points of $\Sigma(\mathcal{F}'_1, \mathcal{F}'_2)$.

• Let \mathbf{p} be a regular point of $\Sigma(\mathcal{F}'_1, \mathcal{F}'_2)$ and let H be the component containing \mathbf{p} . Let's pick local coordinates (u, v) at \mathbf{p} such that $(H, \mathbf{p}) = \{u = 0\}$. Let θ_i be a local generator of \mathcal{F}'_i , for $i = 1, 2$. Thus we find that $\theta_1 \wedge \theta_2 = \text{Unit} \cdot u^m du \wedge dv$, for a positive integer m . Note that H is either invariant for \mathcal{F}'_1 and \mathcal{F}'_2 or is di-critical for \mathcal{F}'_1 and \mathcal{F}'_2 .

Case 1. Suppose $\theta_1(\mathbf{p}) \neq 0$ and $\theta_2(\mathbf{p}) \neq 0$. Then we check that $\theta_2 = \text{Unit} \cdot \theta_1 + u^m \omega$ with ω such that $\omega = dv$ if H is invariant for θ_1 and $\omega = du$ if H is di-critical for θ_1 .

Case 2. Suppose $\theta_1(\mathbf{p}) \neq 0$ and $\theta_2(\mathbf{p}) = 0$. Thus H is invariant for \mathcal{F}'_2 . We get that $\theta_1 = du + uB_1 dv$ while $\theta_2 = (v^k \phi(v) + uA_2) du + uB_2 dv$, for function germs A_2, B_1, B_2 such that $uB_2 - uB_1(v^k \phi(v) + uA_2) = u^m$, so that $\theta_2 = (v^k \phi(v) + uA_2)\theta_1 + u^m dv$. Note that $m = 1$ if and only if B_2 is a unit.

Case 3. Suppose $\theta_1(\mathbf{p}) = u dv + (v^{k_1} \phi_1(v) + uB_1) du$ and $\theta_2(\mathbf{p}) = 0$. Thus H is invariant for both foliations and necessarily $m \geq 2$. We also write $\theta_2 = u(v^{k_2} \phi_2(v) + uB_2) du + uA_2 dv$ so that $(v^{k_2} \phi_2 + uB_2) - A_2(v^{k_1} \phi_1 + uB_1) = u^{m-1}$, thus $\theta_2 = u^{m-1} dv + A_2 \theta_1$.

There would be another case to consider, but the situation in which we will use these normal forms and their behavior is, as we will see in Section 10, not generic. We will deal with this last situation in due time.

• Let \mathbf{p} be a corner point of $\Sigma(\mathcal{F}'_1, \mathcal{F}'_2)$. Let (u, v) be local coordinates centered at \mathbf{p} such that we can write $(\Sigma(\mathcal{F}'_1, \mathcal{F}'_2), \mathbf{p}) = \{uv = 0\}$. Let θ_i be local generator of \mathcal{F}'_i , for $i = 1, 2$. Thus we find that $\theta_1 \wedge \theta_2 = \text{Unit} \cdot u^m v^n du \wedge dv$, for positive integers m and n .

Case 4. Suppose $\theta_1(\mathbf{p}) \neq 0$ and $\theta_2(\mathbf{p}) \neq 0$. Up to permuting u and v , we can find θ_1 and θ_2 such that $\theta_1 = du + u(\dots) dv$ and $\theta_2 = \text{Unit} \cdot \theta_1 + u^m v^n dv$.

Case 5. Suppose $\theta_1(\mathbf{p}) \neq 0$ and $\theta_2(\mathbf{p}) = 0$. We find θ_1 such that $\theta_1 = du + u(\dots) dv$, up to permuting u and v . The point \mathbf{p} is a singularity of \mathcal{F}'_2 adapted to $\Sigma(\mathcal{F}'_1, \mathcal{F}'_2)$ and $\theta_2 = w dz + z(\dots) dw$ for $(w, z) = (u, v)$ or $(w, z) = (v, u)$. We deduce that $\theta_2 = \text{Unit} \cdot v \theta_1 + u^m v^n dv$. The other case is not possible since n must be positive.

Case 6. Suppose $\theta_1(\mathbf{p}) = 0$ and $\theta_2(\mathbf{p}) = 0$. The point \mathbf{p} is a singularity of \mathcal{F}'_1 and of \mathcal{F}'_2 adapted to $\Sigma(\mathcal{F}'_1, \mathcal{F}'_2)$. Up to permuting u and v we find θ_1 such that $\theta_1 = v du + u(\dots) dv$. We know that $\theta_2 = w dz + z(\dots) dw$ for $(w, z) = (u, v)$ or $(w, z) = (v, u)$. We deduce, up to a multiplication by a local unit, we can find θ_2 such that $\theta_2 = \text{Unit} \cdot \theta_1 + u^m v^{n-1} dv$.

7. RESOLUTION OF SINGULARITIES WITH GAUSS REGULAR MAPPING

The material presented here, although part of the known folklore, introduces useful notions and notations. We are very grateful to Pierre Milman for telling us about Gauss regular desingularization and also strongly recommending using Plücker embedding in order to obtain Proposition 7.2, the main result of this section.

Let $\mathbf{G}_k(V)$ be the Grassmann-bundle of k -dimensional real vector subspaces of the finite dimensional real vector space V . We denote by $[P]$ the point of $\mathbf{G}_k(V)$ corresponding to the k -dimensional vector subspace P of V .

Let F be a regular vector bundle of positive finite rank r over a regular manifold N of finite dimension. Let $\mathbf{G}_k(F)$ be the Grassmann bundle of the k -vector subspaces in the fiber of F .

Let M_0 be a connected regular manifold of dimension n .

Let X_0 be a singular sub-variety of the regular manifold M_0 . Let Y_0 be the singular locus of X_0 .

We recall that the Gauss mapping $\nu_{X_0} : X_0 \setminus Y_0 \rightarrow \mathbf{G}(TM_0) := \cup_{k=1}^{\dim M_0} \mathbf{G}_k(TM_0)$ of X_0 is defined as

$$\underline{b}_0 \in X_0 \setminus Y_0 \rightarrow [T_{\underline{b}_0} X_0] \in \mathbf{G}_{\dim(X_0, \underline{b}_0)}(T_{\underline{b}_0} M_0).$$

Definition 7.1. Let $\pi : (X, E) \rightarrow (X_0, Y_0) \hookrightarrow M_0$ be a geometrically admissible resolution of singularities of X_0 . The resolution π is said Gauss regular, if the mapping $\nu_{X_0} \circ \pi$ extends over X as a regular mapping $X \rightarrow \mathbf{G}(TM_0)$.

Composing a geometrically admissible Gauss regular resolution of singularities of X_0 with any geometrically admissible blowing-up with center in the exceptional divisor will yield another Gauss regular resolution of singularities of X_0 .

Proposition 7.2. There exists a Gauss regular resolution of singularities of X_0 .

Proof. For simplicity we suppose that X_0 is of pure dimension d

Let $\tau_1 : (M_1, X_1, E_{M_1}) \rightarrow (M_0, X_0, Y_0)$ be a geometrically admissible embedded resolution of singularities of X_1 . Let σ_1 be the restriction mapping $\tau_1|_{X_1}$ and let $E_1 := X_1 \cap E_{M_1}$ which is a nc-divisor of the resolved manifold X_1 .

Let $F_0(\sigma_1)$ be the \mathcal{O}_{X_1} -ideal sheaf locally generated by the maximal minors of the differential mapping $D\sigma_1$ (the choice of regular local coordinates is irrelevant). Its co-support is exactly the critical locus of σ_1 and is contained in E_1 . Given any geometrically admissible blowing-up β_C with center $C \subset X_1$, it is easy to check that there exists a non-negative integer α (depending on C) such that the ideal $F_0(\sigma_1 \circ \beta_C)$ factors as $(I_{E_C})^\alpha \cdot \beta_C^* F_0(\sigma_1)$, where $E_C := \beta_C^{-1}(C)$ is the newly created exceptional hypersurface and I_{E_C} is its reduced ideal, which is principal. This means that, with further geometrically admissible blowings-up (with centers in E_1), we could already have assumed that $F_0(\sigma_1)$ was principal and monomial in E_1 , which we do.

For any point $\underline{a}_1 \in X_1 \setminus E_1$, we know that $D\sigma_1(\underline{a}_1) \cdot T_{\underline{a}_1} X_1 = T_{\sigma_1(\underline{a}_1)} X_0$. Let \underline{a}_1 be any point of E_1 and let (u, v) be local coordinates adapted to E_1 . Let (u', v') be another system of local coordinates adapted to E_1 . Thus, in a neighborhood \mathcal{U}_1 of a_1 in X_1

$$\begin{aligned} D\sigma_1 \cdot \partial_{u_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{u_s} \wedge D\sigma_1 \cdot \partial_{v_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{v_t} = \\ \text{Unit} \cdot D\sigma_1 \cdot \partial_{u'_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{u'_s} \wedge D\sigma_1 \cdot \partial_{v'_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{v'_t} \end{aligned}$$

where $s + t = d$ is the dimension of X_0 .

Since $F_0(\sigma_1)$ is principal and monomial in E_1 , we see that there exists a nowhere vanishing regular mapping $\gamma_1 : \mathcal{U}_1 \rightarrow \wedge^d TM_0$ such that

$$D\sigma_1 \cdot \partial_{u_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{u_s} \wedge D\sigma_1 \cdot \partial_{v_1} \wedge \cdots \wedge D\sigma_1 \cdot \partial_{v_t} = \mathcal{M}_1 \cdot \gamma_1$$

where \mathcal{M}_1 is a local generator of $F_0(\sigma_1)$. So we deduce that the mapping $\mathcal{U}_1 \ni \underline{a} \rightarrow [\gamma_1(\underline{a})] \in \mathbf{G}_d(TM_0)$ (using here the Plücker embedding of $\mathbf{G}_d(TM_0)|_{\mathcal{U}_0}$, where \mathcal{U}_0 is a neighborhood of $\sigma_1(\underline{a}_1)$ over which TM_0 is trivial) where $[\gamma_1(\underline{a})]$ is the vector space direction corresponding to the d -vector $\gamma_1(\underline{a})$. This regular mapping coincides with $\nu_{X_0} \circ \sigma_1$ on $\mathcal{U}_1 \setminus E_1$.

When X_0 is not of pure dimension, we find that the resolved manifold X_1 is a disjoint union of regular manifolds. Therefore we can proceed exactly as above, independently for each dimension. \square

8. 2-SYMMETRIC TENSORS AND QUADRATIC FORMS ON SINGULAR SUB-VARIETIES

Thanks to Section 7 we will make sense here of the notion of restriction of 2-symmetric tensor, respectively quadratic forms, on singular sub-varieties.

Let V be a real vector space of finite dimension r . The *universal Grassmann bundle* $\tilde{\mathbf{G}}_k(V)$ is the algebraic sub-variety of $\mathbf{G}_k(V) \times V$ consisting of the pairs $([P], v)$ for any vector $v \in P$ with P any k -dimensional vector subspace of V .

Let M_0 be a connected analytic manifold of finite dimension.

Let Z be any non-empty sub-variety of M_0 . Let \mathcal{N}_Z be the Nash "bundle" of Z in M_0 . It is the closed semi-analytic subset of $\mathbf{G}(TM_0)$ consisting of all the (converging) limits of tangent sub-spaces to Z at regular points of Z . Or in other words it is just the (topological) closure of the graph of the Gauss mapping of Z . Let $C_4(Z)$ be the closure, taken into TM_0 , of the tangent bundle TZ_{reg} of the regular part Z_{reg} of Z . We call it the *pseudo-tangent "bundle"* of Z . We denote it by $C_4(Z)$, since point-wise, the fiber $C_4(Z, \underline{a})$ over a point $\underline{a} \in X_0$ is the 4-th Whitney tangent cone [23] and consists of the union $\cup_{[P] \in \mathcal{N}_{\underline{a}}Z} P$ of all the limits at \underline{a} of the tangent spaces to Z at regular points of Z . When Z is a submanifold $C_4(Z)$ is just the usual tangent bundle TZ .

Let X_0 be a singular sub-variety of M_0 with non empty singular locus Y_0 . In order to avoid a not very useful discussion (and further notations), we will define the restriction of a 2-symmetric tensor on M_0 to X_0 , via the polar form, once is presented the notion of restriction of regular quadratic forms on M_0 to X_0 in the next definition. Since $C_4(X_0)$ is a subset of TM_0 we can introduce the following

Definition 8.1. Let κ be a regular quadratic form on M_0 . The restriction of κ to X_0 , denoted $\kappa|_{X_0}$, is defined as the restriction $\kappa|_{C_4(X_0)}$ of κ to the pseudo-tangent "bundle" of X_0 .

The restriction of the 2-symmetric tensor \mathcal{K} on M_0 to X_0 is just defined via the polar form of the restriction of \mathcal{K}_Δ to X_0 .

On the regular part $X_0 \setminus Y_0$ of X_0 this definition coincide with the restriction $X_0 \setminus Y_0 \ni \underline{a} \rightarrow \kappa(\underline{a})|_{T_{\underline{a}}X_0}$, respectively of $X_0 \ni \underline{a} \rightarrow \mathcal{K}(\underline{a})|_{\text{Sym}^2(T_{\underline{a}}X_0)}$.

Let $\tau : (T_0, D_0) \rightarrow X_0$ be any Gauss regular admissible resolution of singularities of X_0 .

Let ν_0 be the regularized Gauss mapping of X_0 , that is the regular mapping $T_0 \rightarrow \mathbf{G}(TM_0)$ extending to the whole of T_0 the parameterized Gauss mapping $\nu_{X_0} \circ \tau : T_0 \setminus D_0 \rightarrow X_0 \setminus Y_0$. We see that $\cup_{\underline{a} \in X_0} \cup_{\underline{b} \in \tau^{-1}(\underline{a})} (\underline{a}, \nu_0(\underline{b})) = \mathcal{N}_{X_0}$.

For any point $\underline{b} \in T_0$, let $T_{\underline{b}}^\tau X_0$ be the vector subspace of $T_{\tau(\underline{b})}M_0$ whose direction is the value at \underline{b} of the regular extension ν_0 , namely $\nu_0(\underline{b}) = [T_{\underline{b}}^\tau X_0] \in \mathbf{G}(T_{\tau(\underline{b})}M_0)$. We call the vector sub-space $T_{\underline{b}}^\tau X_0$ the *tangent space of X_0 at \underline{b} along τ* . We deduce that

$$C_4(X_0) = \cup_{\underline{a} \in X_0} \cup_{\underline{b} \in \tau^{-1}(\underline{a})} \underline{a} \times T_{\underline{b}}^\tau X_0$$

and that for each $\underline{b} \in T_0$, the differential mapping $(D\tau)(\underline{b}) : T_{\underline{b}}T_0 \rightarrow T_{\tau(\underline{b})}M_0$ takes its values in $T_{\underline{b}}^\tau X_0$.

Let $\tilde{\mathbf{G}}_k(TM_0)$ be the universal bundle associated with $\mathbf{G}_k(TM_0)$ and let $\tilde{\mathbf{G}}(TM_0) := \cup_{k=1}^r \tilde{\mathbf{G}}_k(TM_0)$, the corresponding universal bundle. Let $\tilde{\pi} : \tilde{\mathbf{G}}(F) \rightarrow F$, defined as $(\underline{a}, [P], v) \rightarrow (\underline{a}, v)$.

Taking the graph of ν_0 , embedding it in the fibered product $T_0 \times_{M_0} \mathbf{G}(TM_0)$, then lifting it in the fibered product $T_0 \times_{M_0} \tilde{\mathbf{G}}(TM_0)$ and eventually projecting this lift in the fibered product $T_0 \times_{M_0} TM_0$ via the mapping $\tilde{\pi}$ shows that the union

$$T^\tau X_0 := \cup_{\underline{b} \in X_1} T_{\underline{b}}^\tau X_0,$$

called the *tangent bundle of X_0 along τ* , is a regular vector bundle over the resolved manifold T_0 . We observe that outside the critical locus of τ the restricted vector bundle $(T^\tau X_0)|_{T_0 \setminus D_0}$ is just the pull-back $(\tau|_{T_0 \setminus D_0})^* T(X_0 \setminus Y_0)$.

Thanks to Definition 8.1, the restriction of any submodule of $\Gamma_{M_0}(\text{Sym}(2, TM_0))$ to X_0 is well defined.

Suppose given $\pi_1 : (X_1, E_1) \rightarrow (X_0, Y_0)$, a Gauss regular resolution of singularities of X_0 and let \mathcal{L}_0 be an invertible sub-module of $\Gamma_{M_0}(\text{Sym}(2, TM_0))$. Thus the regular "section" $(\pi_1^* \mathcal{L}_0)|_{T^{\pi_1} X_0}$ of $\text{Sym}(2, T^{\pi_1} X_0)$ coincides with $\pi_1^*(\mathcal{L}_0|_{X_0})$. Namely for κ a local generator of \mathcal{L} nearby $\underline{a}_0 \in X_0$ and for any $\underline{a}_1 \in \sigma_1^{-1}(\underline{a}_0)$, we find

$$((\sigma_1^* \kappa)|_{T^{\pi_1} X_0})(\underline{a}_1) = \kappa(\sigma_1(\underline{a}_1))|_{T^{\pi_1} X_0}.$$

Let $\mathcal{L}_1 := (\pi_1^* \mathcal{L}_0)|_{T^{\pi_1} X_0}$ and let $\mathcal{C}_{\mathcal{L}_1}$ be the \mathcal{O}_{X_1} -ideal of coefficients of \mathcal{L}_1 obtained by evaluating the "2-symmetric tensor" \mathcal{L}_1 along the regular section germs of $\text{Sym}^2(T^{\pi_1} X_0)$. With further locally finite geometrically admissible blowings-up we can assume that $\mathcal{C}_{\mathcal{L}_1}$ is principal and monomial in E_1 . Thus any local generator of the invertible \mathcal{O}_{X_1} -submodule $\mathcal{C}_{\mathcal{L}_1}^{-1} \mathcal{L}_1$ does not vanish anywhere.

Remark 5. Suppose X_0 is a surface and that \mathcal{L}_0 is an orientable invertible submodule of $\Gamma_{M_0}(\text{Sym}^2(TM_0))$. If $\pi_1 : X_1 \rightarrow X_0$ is any Gauss regular resolution of singularities of X_0 . Then we check that $(\pi_1^* \mathcal{L}_0)|_{T^{\pi_1} X_0}$ is orientable by density arguments.

9. MAIN RESULT: MONOMIALIZATION OF 2-SYMMETRIC TENSORS ON REGULAR SURFACES

We present here the main result of the paper. We start with some well known facts about morphisms between vector-bundles. At the very end of the section, we will recall the two situations we want to apply the main result to and which were the starting points of the paper.

Let $\sigma : M \rightarrow N$ be a regular mapping between regular manifolds. Any fiber-bundle considered below will be a regular fiber-bundle, unless explicitly mentioned otherwise.

Let F be a vector bundle of finite rank over N . The base change $\sigma : M \rightarrow N$ induces a regular mapping of vector bundle $\sigma_F^h : \sigma^* F \rightarrow F$, induces σ on the 0-sections and identity in the fibers.

If $A : F \rightarrow F'$ is a regular mapping of vector bundles (both of finite rank) over N , the base-change mapping σ induces a regular mapping $\sigma^* A : \sigma^* F \rightarrow \sigma^* F'$ of vector bundles over M .

Let F be a vector bundle over N and E be a vector bundle over M , both of finite rank. Let $\Phi : E \rightarrow F$ be a regular vector bundles mapping along σ , that is such that $\pi_N^F \circ \Phi = \sigma \circ \pi_M^E$, where π_B^E denotes the projection of the vector bundle E onto its basis B .

We can thus define the regular mapping $\Phi^* A := A \circ \Phi : E \rightarrow F'$ of vector bundles along σ .

There exists also a unique regular mapping $\Phi^\sigma : E \rightarrow \sigma^* F$ of regular vector bundles over M factoring Φ through σ_F^h , namely $\sigma_F^h \circ \Phi^\sigma = \Phi$.

The differential mapping $D\sigma : TM \rightarrow TN$ is a regular mapping of vector bundle along σ . Thus it factors as $D\sigma = \sigma_{TN}^h \circ \Delta\sigma$, where $\Delta\sigma := (D\sigma)^\sigma : TM \rightarrow \sigma^* TN$. This allows to pull-back any \mathcal{O}_M -section $\theta : M \rightarrow \sigma^* T^* N$ as the regular \mathcal{O}_M -section $(\Delta\sigma)^* \theta : M \rightarrow T^* M$, in other words, a regular 1-form on M .

Now **confusion due to competitive notations** may arise. Indeed, from the point of view of vector bundles (and modules) notations, given a 1-form θ on N , the notation $\sigma^* \theta$ just means a section $M \rightarrow \sigma^* T^* N$. But the classical notation of differential topology denotes σ^* the regular mapping $T^* N \rightarrow T^* M$ of vector bundles along σ induced by $D\sigma$, so that $\sigma^* \theta$ means a section $M \rightarrow T^* M$.

Since we will pull-back differential forms and sub-modules of differential forms in the vector bundle (or module) sense as well as we will pull-back these differential forms and sub-modules in the sense of differential topology we take the following convention:

Important change of notations. Let θ be a regular differential 1-form over N .

- The notation $\sigma^* \theta$ will just mean the section $\theta \circ \sigma$ in $\Gamma_M(\sigma^* T^* N)$.

- The notation $(D\sigma)^* \theta$ will mean the pull-back of θ in the usual sense of differential topology, that is $(D\sigma)^* \theta = \theta \circ D\sigma \in \Omega_M^1$ (which in Section 5 and Section 6 was - then un-mistakenly - denoted $\sigma^* \theta$).

The relation between these notations being:

$$\theta \circ D\sigma = (D\sigma)^*\theta = (\Delta\sigma)^*(\sigma^*\theta) = (\Delta\sigma)^*(\theta \circ \sigma) = (\theta \circ \sigma) \circ \Delta\sigma.$$

We introduce the following (tailored) notion, in order to deal simultaneously with the two situations we have in mind.

Definition 9.1. *A regular vector bundle B , of finite rank, over a connected regular manifold N is an almost tangent bundle of N , if there exists a regular mapping $\aleph : TN \rightarrow B$ of vector bundles over N and a subvariety $V \subset N$ of codimension at least one, such that $\aleph : TN|_{N \setminus V} \rightarrow B|_{N \setminus V}$ is a regular isomorphism of vector bundles over $N \setminus V$, and V is minimal (for the inclusion) for this property. We call V the fiber-critical locus of \aleph and denote it $V_\aleph := V$. We will write (B, \aleph, V_\aleph) for such an almost tangent bundle structure.*

Although this notion seems artificial it is not so (as we will explain at the end of the section). The conic tangent bundle of [13] is a typical example of this structure.

Let $\sigma : M \rightarrow N$ as above and let (B, \aleph, V_\aleph) be an almost tangent bundle of N . We further assume that $\sigma^{-1}(V_\aleph)$ is everywhere of co-dimension larger than or equal to 1.

Let $\sigma^\bullet \aleph : TM \rightarrow \sigma^* B$ be the mapping of regular vector bundles over M defined as

$$\sigma^\bullet \aleph := (D\sigma)^* \aleph = \aleph \circ D\sigma = (\Delta\sigma)^*(\sigma^* \aleph).$$

Any \mathcal{O}_N -sub-module Θ of $\Gamma_N(B^*)$ is pulled-back as the \mathcal{O}_M -sub-module $(\sigma^\bullet \aleph)^* \Theta$ of Ω_M^1 via $\sigma^\bullet \aleph$.

A last word about the notations: any given 2-symmetric tensor on B , say $\kappa : N \rightarrow \text{Sym}(2, B)$, when pulled-back by $\sigma^\bullet \aleph$, gives rise to a 2-symmetric tensor on M , namely $(\sigma^\bullet \aleph)^* \kappa : M \rightarrow \text{Sym}(2, TM)$, which is to be confused neither with $\sigma^* \kappa$, which is a regular section $N \rightarrow \text{Sym}(2, \sigma^* B)$, nor with $\kappa_\aleph := \aleph^* \kappa$ which is a regular section $N \rightarrow \text{Sym}(2, TN)$. In this latter case, we see that the ideal of coefficients $\mathcal{C}_{\kappa_\aleph}$ of κ_\aleph is contained in \mathcal{C}_κ . If \mathcal{L} is a non-zero invertible \mathcal{O}_N -sub-module of $\Gamma_N(\text{Sym}(2, B))$, the degeneracy locus of $\aleph^* \mathcal{L}$ will always contains V_\aleph .

The next statement refers explicitly to Proposition 4.5 and uses intensively notations introduced in Section 4. We will denote any strict transform of a given nc-divisor Δ by the symbol Δ^{str} , in the exception of exceptional divisor where strict transforms will still be denoted with the same symbol.

Theorem 9.2. *Let S be a regular surface. Let (B, \aleph, V_\aleph) be an almost tangent bundle on S admitting a fiber-metric \mathbf{g} and let \mathcal{L} be a non-zero invertible \mathcal{O}_N -sub-module of $\Gamma_S(\text{Sym}(2, B))$ which is orientable. Suppose that $\sigma_R : (R, E_R) \rightarrow S$ is a locally finite composition of point blowings-up such that Proposition 4.5 holds true for $\mathcal{L}_\aleph := \aleph^* \mathcal{L}$.*

1) *There exists $\beta' : (S', E' := E_{\beta'} \cup E_R) \rightarrow (R, E_R)$, a locally finite composition of point blowings-up where $E_{\beta'}$ is the exceptional divisor of β' , such that denoting $\sigma' := \sigma_R \circ \beta'$, the triple $(B', \aleph', V_{\aleph'}) := (\sigma'^* B, \sigma'^\bullet \aleph, V_{\sigma', \bullet \aleph})$ is an almost tangent bundle of S' and $V_{\aleph'} = E_{\beta'} \cup \sigma'^{-1}(V_\aleph) = E' \cup \sigma'^{-1}(V_\aleph)$ is a nc-divisor which is normal crossing with $V_{\aleph'}^{\text{str}} \cup D_{\aleph'}^{\text{str}} \cup \Delta_{\aleph'}^{\text{str}}$, where $\Delta_{\aleph'} := VD_{\aleph'}$ if \mathcal{L} is not constant along the fibers, and $\Delta_{\aleph'} := V_\Theta$ if \mathcal{L} is constant along the fibers (we recall that $D_{\aleph'} = \emptyset$ once \mathcal{L}_\aleph is not generically non-degenerate).*

2) *There exists $\tilde{\beta} : (\tilde{S}, \tilde{E} := E_{\tilde{\beta}} \cup E') \rightarrow (S', E')$, a locally finite sequence of points blowings-up such that the following statements hold true.*

Let $\tilde{\sigma} := \sigma' \circ \tilde{\beta}$. Let $\tilde{D}_{\aleph'}$ be the nc-divisor $V_{\aleph'} \cup V_{\aleph'}^{\text{str}} \cup D_{\aleph'}^{\text{str}} \cup \Delta_{\aleph'}^{\text{str}}$ where $V_{\aleph'}$ denotes $\tilde{E} \cup \tilde{\sigma}^{-1}(V_\aleph)$.

i) For each $i = 1, 2$, the invertible sub-module $\mathcal{X}_i := (\Delta\tilde{\sigma})^((\beta' \circ \tilde{\beta})^* \Theta_i)$ of $\Omega_{\tilde{S}}^1$ factors as $\mathcal{X}_i = \mathcal{J}_i \cdot \mathcal{F}_i$, where \mathcal{J}_i is a principal and monomial ideal in the nc-divisor $V_{\aleph'} \cup \Delta_{\aleph'}^{\text{str}}$ and \mathcal{F}_i is an invertible sub-module of $\Omega_{\tilde{S}}^1$ only with singularities adapted to $E_{\tilde{\beta}}$. Moreover each foliation \mathcal{F}_i is adapted to the nc-divisor $\tilde{D}_{\aleph'}$.*

ii) Each point $\tilde{\mathbf{p}} \in \tilde{S}$ admits a neighborhood $\tilde{\mathcal{U}}$ of $\tilde{\mathbf{p}}$ in \tilde{S} such that, denoting ω_i a local generator of \mathcal{F}_i and $\kappa_\aleph = \aleph^ \kappa$ for κ a local generator of \mathcal{L} , we find*

$$(8) \quad \tilde{\aleph}^* \kappa = (D\tilde{\sigma})^* \kappa_\aleph = \mathcal{M}_{\aleph'}[\varepsilon_1 \mathcal{N}_1(\mathcal{M}_1 \omega_1) \otimes (\mathcal{M}_1 \omega_1) + \varepsilon_2 \mathcal{N}_2(\mathcal{M}_2 \omega_2) \otimes (\mathcal{M}_2 \omega_2)],$$

where

- Where $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$;
- The germ $\mathcal{M}_{\mathcal{L}_\mathbb{N}}$ is a monomial locally generating $\tilde{\sigma}^* \mathcal{C}_{\mathcal{L}_\mathbb{N}}$.
- For $i = 1, 2$, the germ \mathcal{M}_i is a monomial locally generating \mathcal{J}_i .
- if \mathcal{L} is generically non-degenerate, both function germs $\mathcal{N}_1, \mathcal{N}_2$ are local monomials in $V_\mathbb{N} \cup D_{\mathcal{L}_\mathbb{N}}^{\text{str}}$ which cannot both vanish simultaneously. And one of the monomial $\mathcal{N}_i \mathcal{M}_{\mathcal{L}_\mathbb{N}}$ is a local generator of $\tilde{\sigma}^* I_{\mathcal{L}_\mathbb{N}}^D$.
- if \mathcal{L} is everywhere degenerate one of the function germs $\mathcal{N}_1, \mathcal{N}_2$ is a local monomial in $V_\mathbb{N} \cup D_{\mathcal{L}_\mathbb{N}}^{\text{str}}$ while the other one is identically zero. If \mathcal{N}_i , for $i = 1$ or 2 , is not the zero monomial then $\mathcal{N}_i \mathcal{M}_{\mathcal{L}_\mathbb{N}}$ is a local generator of $\tilde{\sigma}^* I_{\mathcal{L}_\mathbb{N}}^D$.

iii) The sub-module $\mathcal{X}_1 \wedge \mathcal{X}_2$ writes as $\mathcal{J}_{1,2} \cdot \Omega_{\tilde{S}}^2$ where $\mathcal{J}_{1,2}$ is a principal ideal monomial in the nc-divisor $\tilde{\Sigma}_{1,2} := \text{co-supp}(\mathcal{J}_{1,2})$ (containing $V_\mathbb{N} \cup \Delta_{\mathcal{L}_\mathbb{N}}$) which is normal crossing with $\tilde{\mathcal{D}}_{\mathcal{L}_\mathbb{N}}$.

iv) Let $\mathcal{M}_{1,2}^{\log}$ be a local generator of $\mathcal{J}_{1,2}^{\log}$ at $\tilde{\mathbf{p}}$ the ideal of logarithmic coefficients of $\mathcal{X}_1 \wedge \mathcal{X}_2$ as a sub-module of $\Omega_{\tilde{S}}^2(\log(\tilde{\mathcal{D}}_{\mathcal{L}_\mathbb{N}} \cup \tilde{\Sigma}_{1,2}))$. For $i = 1, 2$, let \mathcal{M}_i^{\log} be a local generator of the logarithmic coefficient ideal $\mathcal{C}_{\mathcal{X}_i}^{\log}$ of \mathcal{X}_i as a sub-module of $\Omega_{\tilde{S}}^1(\log(\tilde{\mathcal{D}}_{\mathcal{L}_\mathbb{N}} \cup \tilde{\Sigma}_{1,2}))$. The monomials $(\mathcal{M}_{\mathcal{L}_\mathbb{N}} \mathcal{N}_1 \mathcal{M}_1^{\log})$, $(\mathcal{M}_{\mathcal{L}_\mathbb{N}} \mathcal{N}_2 \mathcal{M}_2^{\log})$, $\mathcal{M}_{1,2}^{\log}$ are ordered.

Before starting the proof, a word on notations. We start with a sub-module Θ_i of regular sections $R \rightarrow (\sigma_R^* B_\mathbb{N})^*$ where $B_\mathbb{N} := \mathbb{N}^* B$. We can pull-back Θ_i on R as a sub-module of Ω_R^1 , denoted $(\Delta \sigma_R)^* \Theta_i$. So that, up to dividing by what is necessary, we have now two possibly singular foliations on S_R . Thus $(D(\beta' \circ \tilde{\beta}))^*((\Delta \sigma_R)^* \Theta_i)$ is the invertible sub-module \mathcal{X}_i of $\Omega_{\tilde{S}}^1$ which (once divided by what is necessary) provides a foliation onto \tilde{S} .

Proof. Point 1) is straightforward.

For $i = 1, 2$, let Θ'_i be $\beta'^* \Theta_i$ and let $\mathbf{g}' := \mathbf{g} \circ \sigma'$ be the fiber-metric on B' . We have again that Θ'_1 and Θ'_2 are both with empty co-support as $\mathcal{O}_{S'}$ -sub-modules of $\Gamma_{S'}(B'^*)$, and are orthogonal for \mathbf{g}' .

Each module $\mathcal{D}_i := (\Delta \sigma')^*(\beta'^* \Theta_i)$ is a non-zero invertible sub-module of $\Omega_{S'}^1$, and factors as $\mathcal{C}_{\mathcal{D}_i} \cdot \mathcal{D}'_i$ where $\mathcal{C}_{\mathcal{D}_i}$ is the coefficient ideal of \mathcal{D}_i and \mathcal{D}'_i is a finite co-dimensional at each point, thus define a (possibly singular) foliation \mathcal{F}'_i on S' . Note that the co-support of $\mathcal{C}_{\mathcal{D}_i}$ is necessarily in $V_{\mathbb{N}'}$, since outside $V_{\mathbb{N}'}$ we are dealing with an isomorphism.

Observation. Let $\gamma : (S'', E'' = E' \cup E_\gamma) \rightarrow (S', E')$, be the blowing-up of the point $\mathbf{p}' \in S'$ and $E_\gamma := \gamma^{-1}(\mathbf{p}')$, the newly created exceptional hypersurface. Let I_{E_γ} be the reduced ideal of E_γ . Thus we observe that $(D\gamma)^*(\mathcal{D}'_i) = I_{E_\gamma}^{k_i} \mathcal{D}''_i$, where k_i is a positive integer and \mathcal{D}''_i is a sub-module of $\Omega_{S''}^1$ which is finite co-dimensional at each point, thanks to Noether Lemma 5.5.

The simple observation above guarantees that there exists a locally finite sequence of points blowings-up $\beta'' : (S'', E'' = E' \cup E_{\beta''}) \rightarrow (S', E')$ such that for each i the sub-module $(D\beta'')^* \mathcal{D}_i$ factors a $J_i \cdot \mathcal{D}''_i$ where J_i is principal and monomial in $E'' \cup V_{\mathbb{N}'}^{\text{str}} = E_{\beta''} \cup V_{\mathbb{N}'}^{\text{str}}$ and \mathcal{D}''_i is an invertible $\mathcal{O}_{S''}$ -sub-module of $\Omega_{S''}^1$ which is finite co-dimensional at each point.

In order to avoid further notations, we can assume that σ' was already such that each ideal $\mathcal{C}_{\mathcal{D}_i}$ is already principal and monomial in $V_{\mathbb{N}'}$, so that each \mathcal{D}'_i was also already defining a foliation \mathcal{F}'_i on S' .

Now we just have to resolve the singularities of \mathcal{F}'_1 and \mathcal{F}'_2 and do further point blowings-up so that each final pulled-back foliation is in a form as good as it can be with some of the given nc-divisors we want to take care of. But up to a locally finite sequence of points blowings-up we can already assumed, thanks to the results of Section 6, that the mapping σ' achieve this. So we get point i). To get the whole of point ii) there is just to carefully track everything we have at the level of Equation (4) of Proposition 4.5 for κ_4 and since $\tilde{\mathbb{N}}^* \kappa = (\Delta \tilde{\sigma})^*(\beta' \circ \tilde{\beta}^* \kappa_4)$, we check we get what is stated.

Now we deal with point iii). Let us denote by Δ' the nc-divisor $V_{\mathbb{N}'} \cup V_{\mathcal{L}_{\mathbb{N}}}^{\text{str}} \cup D_{\mathcal{L}_{\mathbb{N}}}^{\text{str}} \cup \Delta_{\mathcal{L}_{\mathbb{N}}}^{\text{str}}$. Let $\mathcal{J}'_{1,2}$ be the coefficient ideals of $\mathcal{D}_1 \wedge \mathcal{D}_2 = \mathcal{J}'_{1,2} \cdot \Omega_{S'}^2$, and let Σ' be the tangency locus of the foliation \mathcal{F}'_1 and \mathcal{F}'_2 . We can assume, up to further point blowings-up, that $\Lambda' := \text{co-supp}(\mathcal{J}'_{1,2}) = V_{\mathbb{N}'} \cup \Sigma'$ is a nc-divisor which is normal crossing with Δ' and that $\mathcal{J}'_{1,2}$ is also principal and monomial in Λ' . We can assume moreover, up to further point blowings-up, that each local component of $\mathcal{E}' := \Sigma' \cup \Delta'$ is either invariant or di-critical for both foliation $\mathcal{F}'_i, i = 1, 2$.

At a regular point of \mathcal{E}' , each monomial under scrutiny is of the form u^l for u a local coordinate and l a non-negative integer. So they are already ordered. Let Z' be the subset of corner points of \mathcal{E}' . Thus at each point \mathbf{p}' of Z' and for each $i = 1, 2$, each local component of \mathcal{E}' is either invariant or di-critical for \mathcal{F}'_i . This fact is important since the proof of point v) and point vi) of Proposition 6.6 shows that we can always order the "logarithmic" monomials \mathcal{M}_1^{\log} and \mathcal{M}_2^{\log} . Thus, working with the logarithmic 1-forms along the nc-divisor \mathcal{E}' (instead of those along some components of $V_{\mathbb{N}'}$), these logarithmic monomials can be assumed already ordered at any corner point of \mathcal{E}' .

Let us repeat the argument here: Let ω_i be a local logarithmic generator of \mathcal{D}_i so that the pull-back $(D\sigma')^*\omega_i = \mathcal{M}_i^{\log}\theta_i^{\log}$ where θ_i^{\log} is a local logarithmic generator of \mathcal{F}'_i and where $(D\sigma')^*\mathcal{D}_i$ is seen as a sub-module of $\Omega_{S'}^1(\log \mathcal{E}')$. If γ is the blowing-up of the point \mathbf{p}' of Z' , we see that at each corner point of $\gamma^{-1}(\mathcal{E}') \cap \gamma^{-1}(\mathbf{p}')$, we find out that $(D\gamma)^*\theta_i^{\log}$ is indeed a logarithmic generator of the pulled-back foliation $\gamma^*(\mathcal{F}'_i)$, so that a local generator of the ideal of logarithmic coefficients of $(D(\sigma' \circ \gamma))^*\mathcal{D}_i$ is just the pull back by γ of a local generator of the ideal of logarithmic coefficients of $(D\sigma')^*\mathcal{D}_i$. Our problem of comparison of monomial is indeed just a problem of comparing monomials, forgetting about the foliations.

Thus at a point \mathbf{p}' of Z' , there exists a finite sequence of blowings-up $\pi : (S'', E'') \rightarrow (S', E')$ such that the pull-back of the monomials, we were looking to order at \mathbf{p}' , are ordered at each corner point of $\pi^{-1}(\mathbf{p}') \cap \pi^{-1}(\mathcal{E}')$. \square

The à-priori artificial context of Theorem 9.2, mostly caused by the introduction of the notion of almost tangent bundle, is due to find a formulation which we can apply to the two, and not quite identical, following situations below. That is also why instead of working only with the tangent bundle we decided to work on any regular vector bundle of rank 2.

The first situation is when S is a regular surface and $B = TS$ the tangent bundle. Thus κ can be any 2-symmetric (regular) tensor (field) on S , and may be degenerate somewhere (see [9, 11] for semi-positive definite examples).

The second situation motivated the introduction of the notion of almost tangent bundle. Indeed: Suppose the regular surface S resolves the singularities of an embedded surface $S_0 \subset M_0$, a regular manifold, in such a way that it factors through an embedded resolution of the singularities of S_0 such that the resolution mapping $\sigma : S \rightarrow S_0$ is Gauss-regular, which is possible by Proposition 7.2. (Of course we implicitly assume that a surface S_0 has no connected component which are not two-dimensional.) Taking $B := T^\sigma S_0$ and clearly the triple $(B, \sigma^*(D\sigma), E)$ is an almost tangent bundle of S . We take \mathcal{L} as generated by the pull-back of $(D\sigma)^*(\mathcal{K}|_{S_0})$ of any given invertible \mathcal{O}_{M_0} -sub-module \mathcal{K} of $\Gamma_{M_0}(\text{Sym}(2, TM_0))$. As explained in the introduction, we came across such situations when \mathcal{K} is generated by a given regular metric on M_0 [10, 11].

Remark 6. *The result proved above does not depend on the Riemannian metric \mathbf{g}_0 but only on its conformal class, in other word depends only on the invertible \mathcal{O}_{M_0} -sub-module of $\Gamma_{M_0}(\text{Sym}(2, TM_0))$ generated by \mathbf{g}_0 . Indeed, the choice of the geometrically admissible centers we blow-up (to reach our main result) is not affected at any step, if instead of working with \mathbf{g}_0 we were working with a conformal metric, since the only feature of \mathbf{g}_0 we really need to keep track at any time is simply the notion of orthogonality.*

10. LOCAL NORMAL FORMS OF DIFFERENTIALS AND OF THE INNER METRIC ON SINGULAR SURFACES

We finish this paper addressing the primary motivation of this work: describing locally, in a resolved manifold, the pull-back of the inner metric, by the resolution mapping, of an embedded real surface singularity. As a consequence of the previous sections we get a proof of the Hsiang & Pati property for real surfaces which is a bit different from the existing ones [17, 20, 12, 2].

Notations. In Section 5 and Section 6 we used the notations $\sigma^*\theta$ to pull back the differential 1-form $\theta \in \Omega_S^1$ into a differential 1-form of $\in \Omega_{S'}^1$ for a regular mapping $\sigma : S' \rightarrow S$, to mean $\theta \circ D\sigma$. Below and for this whole section this notation, as we explained in Section 9, is replaced by $(D\sigma)^*\theta$.

10.1. Hsiang & Pati property. Let us recall Hsiang & Pati's original result for projective complex normal surface singularities [17, Section III].

Let M be a smooth manifold. Two (Riemannian) metrics \mathbf{g} and \mathbf{h} on M are *quasi-isometric* if there exists a positive constant C such that $C^{-1}\mathbf{h} \leq \mathbf{g} \leq C\mathbf{h}$. They are *locally quasi-isometric* if each point \underline{a} of M admits an open neighborhood \mathcal{U} (of \underline{a}) such that the restricted metrics $\mathbf{g}|_{\mathcal{U}}$ and $\mathbf{h}|_{\mathcal{U}}$ are quasi isometric.

The next result is the main tool used by Hsiang & Pati to get their result. It is a local result in nature.

Lemma 10.1 ([17, Section III]). *Let $(X_0, \mathbf{0})$ be a normal complex isolated surface singularity germ embedded in $(\mathbb{C}^n, \mathbf{0})$. There exists a finite composition of points blowings-up $\sigma : (X, E) \rightarrow (X_0, \mathbf{0})$ such that:*

i) X is a complex manifold of dimension 2 and $E := \sigma^{-1}(\mathbf{0})$, the exceptional divisor of this desingularization of $(X_0, \mathbf{0})$, is a nc-divisor.

ii) Any regular point \underline{a} of E admits a local regular coordinates (u, v) , centered at \underline{a} , such that in this chart $(E, \underline{a}) = \{u = 0\}$ and the resolution mapping writes locally

$$(9) \quad (u, v) \rightarrow (x, y, z) = \sigma(\underline{a}) + (u^{r+1}, u^{r+1}f(u) + u^{r+s+1}v; u^{r+1}g(u) + u^{r+s+1}Z(u, v)) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-2}$$

for non-negative integers r, s , for $f, g \in \mathbb{C}\{u\}$ and Z a regular map germ $(X, \underline{a}) \rightarrow \mathbb{C}^{n-2}$.

iii) Any corner point \underline{a} of E admits a local regular coordinates (u, v) , centered at \underline{a} , such that in this chart $(E, \underline{a}) = \{uv = 0\}$ and the resolution mapping writes locally

$$(10) \quad (u, v) \rightarrow (x, y, z) = \sigma(\underline{a}) + (u^m v^n, u^m v^n f(u, v) + u^p v^q; u^m v^n g(u, v) + u^p v^q Z(u, v)) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n-2}$$

for non-negative integers $p \geq m$ and $q \geq n$ such that $np - qm \neq 0$ and function germs $f, g, Z \in \mathcal{O}_{\underline{a}}$ such that $df \wedge d(u^m v^n) = dg \wedge d(u^m v^n) \equiv 0$.

We will call such local coordinates (u, v) , in either case, *Hsiang & Pati coordinates*. The corollary of such systematic local presentation of the resolution mapping is Hsiang & Pati result of interest to us, which can be formulated in the following way:

Theorem 10.2 ([17]). *Let X_0 be a normal complex surface singularity germ embedded in $\mathbf{P}\mathbb{C}^n$. Let \mathbf{g}_{X_0} be the restriction to the regular part of X_0 of the Fubini-Study metric on $\mathbf{P}\mathbb{C}^n$. There exists a finite composition of points blowings-up $\sigma : (X, E) \rightarrow X_0$ resolving the singularities of X_0 such that*

i) Each point \underline{a} of E admits Hsiang & Pati coordinates (u, v) like in Lemma 10.1.

ii) When \underline{a} is a regular point of E , the (regular extension of the) pulled-back metric $\mathbf{g}_{X_0} \circ D\sigma|_{\mathcal{U}}$ is quasi isometric to the metric over \mathcal{U} given by

$$du^{r+1} \otimes \overline{du^{r+1}} + du^{r+s+1}v \otimes \overline{du^{r+s+1}v}.$$

iii) When \underline{a} is a corner point of E , the (regular extension of the) pulled-back metric $\mathbf{g}_{X_0} \circ D\sigma|_{\mathcal{U}}$ is quasi isometric to the metric over \mathcal{U} given by

$$du^m v^n \otimes \overline{du^m v^n} + du^p v^q \otimes \overline{du^p v^q}.$$

10.2. Preliminaries for local normal forms.

Let M_0 be a regular connected manifold equipped with a regular Riemannian metric g_0 . Let X_0 be a subvariety with no connected component of dimension other than 2.

Suppose given a Gauss regular resolution $\sigma_1 : (X_1, E_1) \rightarrow X_0$.

Notation. Let $\Omega_{\sigma_1}^1$ be the \mathcal{O}_{X_1} -dual to $\Gamma_{X_1}(T^{\sigma_1}X_0)$. It is a locally free \mathcal{O}_{X_1} -module of rank 2

A differential 1-form along σ_1 is a regular section $X_1 \rightarrow (T^{\sigma_1}X_0)^*$.

Following Proposition 7.2 and then using Proposition 4.5, we can further assume that for any point $\underline{a}_1 \in X_1$ there exists a neighborhood \mathcal{U}_1 of \underline{a}_1 such that there exist local regular sections $\omega_1, \omega_2 \in \Omega_{\sigma_1}^1|_{\mathcal{U}_1}$, with kernels orthogonal (for the fiber-metric g^{σ_1} restriction of $\sigma_1^*g_0$ to $T^{\sigma_1}X_0$), such that over \mathcal{U}_1 the following holds true:

$$(11) \quad g^{\sigma_1} = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2.$$

By definition g^{σ_1} is also the fiber metric onto $T^{\sigma_1}X_0$ which extends the fiber-metric $\sigma_1^*(g_0|_{X_0 \setminus Y_0})$ of $T^{\sigma_1}X_0|_{X_1 \setminus E_1} = \sigma_1^*T(X_0 \setminus Y_0)$.

Suppose given a resolution of singularities $\tilde{\pi} : (\tilde{X}, \tilde{E}) \rightarrow X_0$ like in Theorem 9.2 factoring through σ_1 , that is $\tilde{\pi} = \sigma_1 \circ \tilde{\beta}$ for $\tilde{\beta} : (\tilde{X}, \tilde{E}) \rightarrow (X_1, E_1)$ a locally finite sequence of points blowings-up. Thus Equation (11) becomes

$$(12) \quad g^{\tilde{\pi}} := \tilde{\beta}^*(g^{\sigma_1}) = g^{\sigma_1} \circ \tilde{\beta} = \theta_1 \otimes \theta_1 + \theta_2 \otimes \theta_2,$$

and where $\theta_i := \omega_i \circ \tilde{\beta}$ is a differential 1-form along $\tilde{\pi}$ for $i = 1, 2$. By definition $g^{\tilde{\pi}}$ extends to the whole of $T^{\tilde{\pi}}X_0$ the fiber-metric $\tilde{\pi}^*(g_0|_{X_0 \setminus Y_0})$ of $T^{\tilde{\pi}}X_0|_{\tilde{X} \setminus \tilde{E}}$.

We recall, given a regular mapping $\sigma : M \rightarrow N$, that in Section 9 we factored the differential mapping $D\sigma : TM \rightarrow TN$ as $D\sigma = \sigma_{TN}^h \circ (\Delta\sigma)$, where $\Delta\sigma$ is the regular mapping $TM \rightarrow \sigma^*TN$ of regular vector bundle over M .

From Theorem 9.2, each $\tilde{a} \in \tilde{X}$ admits a neighborhood $\tilde{\mathcal{U}}$ such that for $i = 1, 2$, there exists $\mu_i \in \Omega_{\tilde{\mathcal{U}}}^1$, only with singularities adapted to \tilde{E} , such that

$$(D\tilde{\beta})^*((\Delta\sigma_1)^*\theta_i) = \mathcal{M}_i\mu_i$$

with \mathcal{M}_i a monomial in \tilde{E} , and $\Delta\tilde{\pi} : T\tilde{X} \rightarrow T^{\tilde{\pi}}X_0 \subset \tilde{\pi}^*TM_0$. Let χ_1, χ_2 be local regular sections $\tilde{X} \rightarrow T^{\tilde{\pi}}X_0$, which are orthogonal for the fiber-metric $g^{\tilde{\pi}}$ on $T^{\tilde{\pi}}X_0$, and such that $\theta_i(\chi_j) = \delta_{i,j}$ for $i, j \in \{1, 2\}$. Suppose that $\tilde{\mathcal{U}}$ is small enough such that we can choose regular coordinates (u, v) , centered at \tilde{a} and adapted to \tilde{E} , i.e. $(E, \tilde{a}) \subset \{uv = 0\}$.

Let Q be the matrix of the mapping $(\Delta\sigma_1) \circ D\tilde{\beta}$ in the basis (∂_u, ∂_v) and (χ_1, χ_2) . Let $\text{adj}(Q)$ be the adjoint matrix of Q so that $\text{adj}(Q) \cdot Q = Q \cdot \text{adj}(Q) = \psi\mathcal{M} \cdot Id$, where \mathcal{M} is a monomial in \tilde{E} and ψ and analytic unit over $\tilde{\mathcal{U}}$. Note that $\psi\mathcal{M}$ is the (oriented) volume of the image of $\tilde{\pi}$ and by Theorem 9.2, we have $\mathcal{M} = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_{1,2}$.

We obtain two regular vector fields on \tilde{X} , namely $\zeta_i := \text{adj}(Q)\chi_i$, for $i = 1, 2$. They may vanish only on \tilde{E} . We deduce that $\mathcal{M}_i\mu_i(\zeta_j) = \theta_i(\mathcal{M}\chi_j) = \psi\mathcal{M}\delta_{i,j}$ for $i, j = 1, 2$. Writing $\mu_i = a_i du + b_i dv$ and $\zeta_i = \alpha_i \partial_u + \beta_i \partial_v$ we observe that

$$(13) \quad a_i \alpha_j + b_i \beta_j = \psi \mathcal{M}_k \mathcal{M}_{1,2} \delta_{i,j} \text{ with } i \neq k \text{ and } i, j, k \in \{1, 2\}$$

$$(14) \quad a_1 b_2 - a_2 b_1 = \psi \mathcal{M}_{1,2}$$

Since μ_1 and μ_2 may only vanish at isolated points, there exists real analytic meromorphic functions germs f_1 and f_2 such that

$$(\alpha_i, \beta_i) = f_j \cdot (b_j, -a_j) \text{ with } i \neq j$$

yielding $f_j = \psi \mathcal{M}_j$. In other words, in the basis above the mapping $(\Delta\sigma_1) \circ D\tilde{\beta}$ over $\tilde{\mathcal{U}}$ writes as

$$(\Delta\sigma_1) \circ D\tilde{\beta} = (f_1 \mu_1, f_2 \mu_2) : T_{\tilde{b}}\tilde{X} \ni \xi \rightarrow (f_1 \mu_1(\xi), f_2 \mu_2(\xi)) = (\theta_1, \theta_2)([(\Delta\sigma_1) \circ D\tilde{\beta}] \cdot \xi) \in T_{\tilde{b}}^{\tilde{\pi}}X_0.$$

Note also that along the way, we have proved the following expected

Lemma 10.3. *The $\mathcal{O}_{\tilde{a}}$ -module $(\Delta\tilde{\pi})^*(\Omega_{\sigma_1, \tilde{a}}^1)$ is generated by $\mathcal{M}_1\mu_1$ and $\mathcal{M}_2\mu_2$.*

10.3. Local normal form of differentials.

Let \tilde{a} be a point of \tilde{E} . Let (u, v) be local coordinates centered at \tilde{a} adapted to \tilde{E} , so that $\{u = 0\} \subset (\tilde{E}, \tilde{a}) \subset \{uv = 0\}$.

The $\mathcal{O}_{\tilde{X}}$ -module $\tilde{\pi}^*(\Omega_{M_0}^1)$ is locally free of rank n and we have $(D\tilde{\pi})^*(\Omega_{M_0}^1) = (\Delta\tilde{\pi})^*(\tilde{\pi}^*(\Omega_{M_0}^1))$, which is an $\mathcal{O}_{\tilde{X}}$ -sub-module of $\Omega_{\tilde{X}}^1$ locally free of rank 2.

We recall that $T^{\tilde{\pi}}X_0$ is a vector sub-bundle of $\tilde{\pi}^*TM_0$. Let $\underline{a}_0 := \tilde{\pi}(\tilde{a})$ be the image of \tilde{a} . For a germ of differential form $\theta \in \Omega_{M_0, \underline{a}_0}^1$, let $\theta^{\tilde{\pi}}$ be the local section, nearby \tilde{a} , of $\tilde{X} \rightarrow T^{\tilde{\pi}}X_0$, define as the restriction $(\tilde{\pi}^*\theta)|_{T^{\tilde{\pi}}X_0}$. Let $\Lambda_{\tilde{\pi}} := \tilde{\pi}^*(\Omega_{M_0}^1)|_{T^{\tilde{\pi}}X_0}$ be the $\mathcal{O}_{\tilde{X}}$ -sub-module of $\Omega_{\tilde{X}}^1$ generated by the restrictions to $T^{\tilde{\pi}}X_0$.

Claim 1: $(D\tilde{\pi})^*(\Omega_{M_0}^1) = (\Delta\tilde{\pi})^*\Lambda_{\tilde{\pi}}$.

Proof of the Claim 1. Any germ at \tilde{a} of a vector field ξ induces the germ at \tilde{a} of the local section $D\tilde{\pi} \cdot \xi : (\tilde{X}, \tilde{a}) \rightarrow T^{\tilde{\pi}}X_0$. For $\theta \in \Omega_{M_0, \underline{a}_0}^1$, we get $(D\tilde{\pi})^*\theta \in \Omega_{\tilde{X}, \tilde{a}}^1$ and for every \tilde{b} nearby \tilde{a} , the linear form $((D\tilde{\pi})^*\theta)(\tilde{b})$ is defined as

$$T_{\tilde{b}}\tilde{X} \ni \xi \rightarrow (\theta(\tilde{\pi}(\tilde{b})))((D\tilde{\pi})(\tilde{b}) \cdot \xi),$$

while the linear form $((\Delta\tilde{\pi})^*\theta^{\tilde{\pi}})(\tilde{b})$ is defined as

$$T_{\tilde{b}}\tilde{X} \ni \xi \rightarrow (\theta(\tilde{\pi}(\tilde{b}))|_{(T^{\tilde{\pi}}X_0)_{\tilde{b}}})((D\tilde{\pi})(\tilde{b}) \cdot \xi),$$

so that they coincide since $(D\tilde{\pi})(\tilde{b}) \cdot \xi$ lies in $(T^{\tilde{\pi}}X_0)_{\tilde{b}} \subset T_{\tilde{\pi}(\tilde{b})}M_0$. \square

Claim 2: $\Lambda_{\tilde{\pi}} = \Omega_{\tilde{X}}^1$.

Proof of the Claim 2. We just need to show that $(\Delta\tilde{\pi})^*\Omega_{\tilde{X}}^1 \subset (D\tilde{\pi})^*(\Omega_{M_0}^1)$. Let θ_1, θ_2 as in Equation (12) and let χ_1, χ_2 be the dual basis (for the fiber metric $\mathbf{g}^{\tilde{\pi}}$). Let $\omega_1, \omega_2 \in \Omega_{M_0, \underline{a}_0}^1$ such that $\omega_i(\underline{a}_0)(\chi_j(\tilde{a})) = \delta_{i,j}$. Thus the sections $\omega_1^{\tilde{\pi}}$ and $\omega_2^{\tilde{\pi}}$ are linearly independent nearby \tilde{a} , and the claim is proved. \square

Combining Claim 1 and Claim 2 with Lemma 10.3 yields the following important

Proposition 10.4. *The $\mathcal{O}_{\tilde{X}}$ -module $(D\tilde{\pi})^*\Omega_{M_0}^1$ is locally generated at \tilde{a} by $\mathcal{M}_1\mu_1$ and $\mathcal{M}_2\mu_2$.*

Let (x, y, z_3, \dots, z_n) be local regular coordinates centered at the image point $\underline{a}_0 := \tilde{\pi}(\tilde{a})$. Obviously $(D\tilde{\pi})^*(\Omega_{M_0}^1)$ is $\mathcal{O}_{\tilde{a}}$ -generated by $(D\tilde{\pi})^*dx, (D\tilde{\pi})^*dy, (D\tilde{\pi})^*dz_3, \dots, (D\tilde{\pi})^*dz_n$. Since it is of local rank 2, we can assume that the coordinates at \underline{a}_0 were such that it is generated by $(D\tilde{\pi})^*dx, (D\tilde{\pi})^*dy$. Proposition 10.4 implies, up to a linear change in x and y , that nearby \tilde{a} the following relations hold:

$$(15) \quad \text{Unit} \cdot \mathcal{M}_1\mu_1 = (D\tilde{\pi})^*dx + A(D\tilde{\pi})^*dy$$

$$(16) \quad \text{Unit} \cdot \mathcal{M}_2\mu_2 = B(D\tilde{\pi})^*dx + (D\tilde{\pi})^*dy$$

for $A, B \in \mathbf{m}_{\tilde{a}}\mathcal{O}_{\tilde{a}}$.

- Assume that \tilde{a} is a smooth point of \tilde{E} .

We start with the following obvious

Lemma 10.5. *Since $\mu_1 \wedge \mu_2 = \text{Unit} \cdot \mathcal{M}(du \wedge dv)$, if $\mu_i(\tilde{a}) = 0$ and $\mu_j(\tilde{a}) \neq 0$ when $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$, then $\mu_j = du + u(\dots)dv$ and $\mathcal{M} = u^{1+t}$ for some non-negative integer t .*

When $\underline{\tilde{a}}$ is a smooth point of \tilde{E} , we find $\mathcal{M}_1 = u^r$ and $\mathcal{M}_2 = u^s$ with $s \geq r \geq 0$. Let us write $\mu_i = a_i du + b_i dv$, and x_w for $\partial_w(\tilde{\pi}^*x)$ and y_w for $\partial_w(\tilde{\pi}^*y)$, where w is either u or v . From Equations (15) and (16) we find the following relations:

$$(17) \quad x_u + Ay_u = u^r \psi_1 a_1$$

$$(18) \quad Bx_u + y_u = u^r \psi_2(u^{s-r} a_2)$$

$$(19) \quad x_v + Ay_v = u^r \psi_1 b_1$$

$$(20) \quad Bx_v + y_v = u^r \psi_2(u^{s-r} b_2)$$

where each ψ_i is a local unit.

We deduce that $x = a_0 + u^{1+r}X(u, v)$ and $y = b_0 + u^{1+r}Y(u, v)$ for constants a_0, b_0 . So we can write

$$\begin{aligned} X &= x_0 + vx_1(v) + ux_2(u) + uvx_3(u, v) \\ Y &= y_0 + vy_1(v) + uy_2(u) + uvy_3(u, v) \end{aligned}$$

We are using this local description of the blowing-up mapping to obtain the following possible local forms.

Proposition 10.6. *Assume the point $\underline{\tilde{a}}$ is a regular point of \tilde{E} .*

1) *If $\mu_1(\underline{\tilde{a}}) \neq 0$, then we find $\mu_1 = du + u(\dots)dv$.*

2) *Suppose $\mu_1(\underline{\tilde{a}}) = 0$ and write $\mu_1 = (v^k \phi(v) + uc_1(u, v))du + uDdv$, with $k = 1$ and $\phi(0) \neq 0$ if $D(\underline{\tilde{a}}) = 0$.*

We are in one of the situations listed below:

i) *Suppose $k = 1$ and $D(\underline{\tilde{a}}) \neq 0$. We can choose the local regular coordinates (u, v) centered at $\underline{\tilde{a}}$ and adapted to \tilde{E} such that $x = a_0 + u^{r+1}v$ and $y = b_0 \pm u^{r+1}$ and thus $r = s$ and $t = 1$. Moreover we find out that $\mu_2 = du + u(\dots)dv$.*

ii) *If $k = 1$ and $D(\underline{\tilde{a}}) = 0$, then $r = s$ and $\mu_2 = du + u(\dots)dv$.*

iii) *If $D(\underline{\tilde{a}}) \neq 0$ and $k \geq 2$, then conclusion of point i) hold true.*

Proof. 1) Since $u^{r+1}(X_v + AY_v) = u^r \psi_1 b_1$, we get $b_1 = uc_1$ for some $c_1 \in \mathcal{O}_{\underline{\tilde{a}}}$.

2) Suppose that $\mu_1(\underline{\tilde{a}}) = 0$.

i) Assume that $\mu_1 = u dv + (v\phi + uc_1)du$ with ϕ a local analytic unit such that $-\phi(0) \notin \mathbb{Q}_{>0}$. Since $a_1(\underline{\tilde{a}}) = 0$ we find that $x_0 = 0$. From Equation (19), we find $X_v + AY_v = \psi_1$ with $A(\underline{\tilde{a}}) = 0$, and we see that $x_1(0) \neq 0$. Let $\bar{v} := X(v, u)$. Thus $(u, v) \rightarrow (u, X(u, v))$ is a regular change of coordinates so that $x = a_0 + u^{r+1}\bar{v}$. Since $v = \bar{v}z_1(\bar{v}) + uz_2(u, \bar{v})$, with $z_1(0) \neq 0$, we deduce that $\mu_1 = \text{Unit}[u d\bar{v} + (\bar{v}\bar{\phi}(\bar{v}) + u\bar{c}_1)du]$ with $\bar{\phi}(0) \neq 0$.

Suppose the coordinates (u, v) , centered at $\underline{\tilde{a}}$ and adapted to \tilde{E} , are also such that $\mu_1 = u dv + (v\phi(v) + uc_1)du$ and $x = a_0 + u^{r+1}v$ with $-\phi(0) \notin \mathbb{Q}_{\geq 0}$. Since $\psi_1 \cdot u^r \mu_1 = dx + A dy$, we get

$$\psi_1 \cdot [u dv + (v\phi + uc_1)du] = [(r+1)vdu + u dv] + A[(r+1)Y + uY_u]du + uY_v dv$$

with $A(\underline{\tilde{a}}) = 0$ and ψ_1 a local analytic unit. Thus we find

$$\begin{aligned} \psi_1 &= 1 + uAY_v \\ \psi_1(v\phi + uc_1) &= u + A[(r+1)Y + uY_u] \end{aligned}$$

We deduce that $y_0 \neq 0$, so that Y is a local analytic unit, and thus $y = b_0 + u^{r+1}Y$. Let ε be the sign of y_0 . Let $\bar{u}(u, v) = u(\varepsilon Y)^{\frac{1}{r+1}}$. The change of coordinates $(u, v) \rightarrow (\bar{u}, v)$ is regular, centered at $\underline{\tilde{a}}$ and adapted to \tilde{E} , and we have $y = b_0 + \varepsilon \bar{u}^{r+1}$. Thus $x = a_0 + \zeta(\bar{u}, v)\bar{u}^{r+1}v$ for a local analytic unit ζ . Thus taking $\bar{v} := v\zeta$, we have found local coordinates centered at $\underline{\tilde{a}}$ adapted to \tilde{E} and such that $x = a_0 + \bar{u}^{r+1}\bar{v}$ and $y = b_0 + \varepsilon \bar{u}^{r+1}$ so that $r = s$. This implies that $t = 1$, which can only occur if $\mu_2(\underline{\tilde{a}}) \neq 0$ (otherwise $t \geq 2$). Since $r = s$ and $\mu_2(\underline{\tilde{a}}) \neq 0$, we deduce from point 1) that $\mu_2 = \text{Unit}(d\bar{u} + \bar{u}\bar{c}_2 d\bar{v})$.

ii) Assume that $\mu_1 = (v + uc_1)du + uDdv$ with $D(\underline{\tilde{a}}) = 0$. Let us write $A = vA_1(v) + u(\dots)$. Equation (17) provides

$$(21) \quad (1+r)[x_0 + v(x_1(v) + [y_0 + vy_1(v)]A_1(v)) + u(\dots)] = [v\psi_1(0, v) + u(\dots)].$$

Thus we deduce that $x_0 = 0$ and $(1+r)v(x_1(v) + y_0A_1(v)) = v\psi_1(0, v)$. Thus $x_1(v) + y_0A_1(v)$ is an analytic unit. Since $D(\underline{a}) = A(\underline{a}) = 0$ and by Equation (19) we get

$$(22) \quad X_v + AY_v = \psi_1 \cdot D.$$

We deduce that $x_1(0) = 0$ so that $y_0 \neq 0$. Up to a change of coordinates as in i), we can assume that $Y = b_0 \pm u^{r+1}$. Equation (18) reads

$$(23) \quad B[(r+1)X + uX_u] + (r+1)Y + uY_u = \psi_2 u^{s-r} a_2,$$

and provides $a_2(\underline{a}) \neq 0$ and $r = s$. Tanks to this latter condition we are back in point 1) by permuting μ_1 and μ_2 so that $\mu_2 = du + u(\dots)dv$.

iii) Suppose $\mu_1 = (v^{k+2}\phi(v) + uc_1)du + u dv$ with $k \geq 0$ so that $D \equiv 1$. From Equation (19), we deduce that $x_1(0) = 1$. Adapting Equation (21) to our situation we get $(x_1 + y_0A_1)(0) = (v^{k+1}\psi_1(0, v))(0) = 0$, so that $y_0A_1(0) = -x_1(0) = 1$. So we have $x_1(0) \neq 0, y_0 \neq 0$, thus we reach, after two changes of variables (one to change u and the next one to change v , the same conclusion as i). \square

Remark 7. *An obvious, but unexpected, consequence of Proposition 10.6 is that any regular point of \tilde{E} cannot be a simultaneous singular point of both foliations \mathcal{F}_1 and \mathcal{F}_2 . Moreover according to our notations, at any regular point \underline{a} of \tilde{E} we can assume that we always have $\mu_1(\underline{a}) \neq 0$.*

We now uses these pairs of normal forms to obtain the following Hsiang & Pati type result

Proposition 10.7. *Let \underline{a} be a regular point of \tilde{E} .*

There exist local coordinates (u, v) centered at \underline{a} and adapted to \tilde{E} ($= \{u = 0\}$) such that

1) If $\mu_2(\underline{a}) \neq 0$, then the module $(D\tilde{\pi})^\Omega_{M_0}^1$ is locally generated at \underline{a} by $d(u^{r+1})$ and $d(u^{s+1+m}v)$ for a non-negative integer m .*

2) If $\mu_2 = a_2 du + u dv$ with $a_2(\underline{a}) = 0$, then $t = 1$ and the module $(D\tilde{\pi})^\Omega_{M_0}^1$ is locally generated at \underline{a} by $d(u^{r+1})$ and $d(u^{s+1}v)$.*

3) If $\mu_2 = (v + uc_2)du + ue_2 dv$ with $e_2(\underline{a}) = 0$, then $t \geq 2$ and the module $(D\tilde{\pi})^\Omega_{M_0}^1$ is locally generated at \underline{a} by $d(u^{r+1})$ and $d(u^{s+t}v)$.*

Proof. For simplicity let $\Theta := (D\tilde{\pi})^*\Omega_{M_0}^1$. We recall also that we have $\mu_1 \wedge \mu_2 = Unit \cdot u^t du \wedge dv$.

By Proposition 10.6 we find $\mu_1 = du + uc_1 dv$. Equation (17) gives $x_0 \neq 0$. So that up to an adapted change of coordinates in u , we can assume that $x = a_0 \pm u^{r+1}$. Up to replacing y by $y \pm y_0 x$, we can assume that $y_0 = 0$.

1) Suppose $\mu_2(\underline{a}) \neq 0$.

If $t = 0$, then we can assume that $\mu_2 = dv$. Equation (20) provides $Y_v = \psi_2 u^{s-r}$, so that up to changing v by $Unit \cdot v$, we can assume $y = b_0 + u^{r+2}y_1(u) + u^{s+1}v$. So we get the desired result in this case.

If $t \geq 1$ and $\mu_2(\underline{a}) \neq 0$ then we deduce $\mu_2 = du + (uc_1 + Unit \cdot u^t)dv$. And we still have $\mu_2 = u dv + a_2 du$ but only $\mu_1 = du + u(\dots)dv$. Writing $Y = uy_1(u) + u^p v Z(u, v)$ for a non negative integer p and with Z such that $Z(0, v) \neq 0$, we deduce from Equations(19) and (20)

$$AY_v = u^p A(Z + vZ_v) = \psi_1 c_1 \text{ and } Y_v = \psi_2 u^{s-r} (c_1 + u^{t-1} \psi_3)$$

where ψ_3 is a local unit. Since $A(\underline{a}) = 0$, we deduce $Z + vZ_v = Unit \cdot u^{s+t-r-1-p}$ so that $p = s - r + t - 1$. Thus up to changing v by $Unit \cdot v$, we can assume $y = b_0 + u^{r+2}y_1(u) + u^{s+t}v$. So we also get the desired result in this case.

2) Assume $\mu_2 = u dv + a_2 du$ with $a_2(\underline{a}) = 0$. Thus $t = 1$ and from Equation (20) we get $Y_v = \psi_2 u^{r-s}$. So that after a change of coordinates in v we can assume that $y = b_0 + u^{r+2}y_1(u) + u^{s+1}v$. And we find the announced result.

3) Suppose $\mu_2 = (v + uc_2)du + ue_2dv$ with $e_2(\underline{\tilde{a}}) = 0$. Thus we deduce $t \geq 2$. Since $\mu_1 = du + uc_1dv$ we deduce that $\mu_2 = (\cdots)\mu_1 + u^t dv$. The module Θ is generated by $u^r \mu_1$ and $u^{s+t} dv$. We will use the following

Lemma 10.8. *Suppose $\mu_1 = du + u^p F dv$, with $p \geq 1$ so that $F(0, v) \neq 0$. There exists a change of coordinates $(w, v) \rightarrow (w + w^p \alpha(v), v)$ such that $\alpha(0) = 0$ and $\mu_1 = \text{Unit}(dw + w^{p+1}(\cdots)dv)$.*

Proof. Let $C = c_0(v) + uC_1(u, v)$, so that $c_0(v)$ is not identically 0. Let $u = w + w^p \alpha(v)$. Since $du = [1 + pw^{p-1}\alpha(v)]dw + w^p \alpha'(v)$. We get $\mu_1 = [1 + pw^{p-1}\alpha(v)]dw + w^p(\alpha'(v) + c_0(v))dv + w^{p+1}\beta(v, w)dv$ with $\beta \in \mathcal{O}_{\underline{\tilde{a}}}$. Taking α the primitive of c_0 vanishing at $v = 0$ provides the result. \square

Remark 8. *The equation we solve in $W := 1 + w^{p-1}\alpha(v)$ in the proof of the Lemma admits a formal solution (that is a solution in the real formal power series in two variables), so that up to a formal change of coordinates $u = \bar{u}W(\bar{u}, v)$ for W a formal power series and a unit, we would find $\mu_1 = d\bar{u}$.*

Thanks to the Lemma, used finitely many times, we deduce (despite these changes of coordinates) that Θ is generated by $u^r du$ and $u^{s+t} dv$ which is the result. \square

Using Hsiang & Pati proof (and caring with the fact that -1 has no real square root), with a few elementary computations, we deduce from Proposition 10.7 the existence of Hsiang & Pati coordinates:

Corollary 10.9 (see also [2]). *There exist adapted coordinates (u, v) at the regular point $\underline{\tilde{a}}$ such that the resolution mapping $\tilde{\pi}$ locally writes*

$$(u, v) \rightarrow (x, y, z) = \tilde{\pi}(\underline{\tilde{a}}) + (\pm u^{k+1}, u^{k+1}f(u) + u^{l+1}v; u^{k+1}z(u) + u^{l+1}vZ(u, v)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$$

for non-negative integers k, l , for $(f; z) \in \mathbb{C}\{u\} \times \mathbb{C}\{u\}^{\dim M_0 - 2}$ and Z a regular map germ $(X, \underline{a}) \rightarrow \mathbb{R}^{n-2}$.

• We assume now that $\underline{\tilde{a}}$ is a corner point of \tilde{E} .

If $\underline{\tilde{a}}$ is a regular point of μ_i , for $i = 1$ or 2 , then we can write $\mu_i = dz + z(\cdots)dw$ and where $(w, z) = (u, v)$ or (v, u) . In this case the logarithmic local generator of \mathcal{F}_i writes

$$\mu_i^{\log} = z^{-1}\mu_i = d_{\log}z + w(\cdots)d_{\log}w.$$

If $\underline{\tilde{a}}$ is a singular point of μ_i (thus adapted to \tilde{E}), for $i = 1$ or 2 , then $\mu_i = wdz + z(\cdots)dw$ and where $(w, z) = (u, v)$ or (v, u) . The logarithmic local generator of \mathcal{F}_i writes in this case as

$$\mu_i^{\log} = w^{-1}z^{-1}\mu_i = d_{\log}z + (\cdots)d_{\log}w.$$

By Proposition 10.4, the sub-module $(D\tilde{\pi})^*\Omega_{M_0, \tilde{\pi}(\underline{\tilde{a}})}^1$ of $\Omega_{\tilde{X}, \underline{\tilde{a}}}^1$ is generated over $\mathcal{O}_{\underline{\tilde{a}}}$ by $\mathcal{M}_1\mu_1$ and $\mathcal{M}_2\mu_2$. Thus $(D\tilde{\pi})^*\Omega_{M_0}$, as a $\mathcal{O}_{\tilde{X}}$ -sub-module of $\Omega_{\tilde{X}}^1(\log \tilde{E})$ is locally generated at $\underline{\tilde{a}}$ by $\mathcal{M}_1^{\log}\mu_1^{\log}$ and $\mathcal{M}_2^{\log}\mu_2^{\log}$.

We know that $\mathcal{M}_1^{\log} = u^m v^n$ and $\mathcal{M}_2^{\log} = u^p v^q$ with $\mu_1^{\log} \wedge \mu_2^{\log} = \text{Unit} \cdot u^r v^s (d_{\log}u \wedge d_{\log}v)$. We can assume by Theorem 9.2 that $p \geq m \geq 0, q \geq n \geq 0$ and $m + n \geq 1$. When μ_1 or μ_2 vanishes at $\underline{\tilde{a}}$, we deduce $\max(r, s) \geq 1$. Since the local coordinates (u, v) are centered at $\underline{\tilde{a}}$ and adapted to \tilde{E} , for $i = 1, 2$ we can write $\mu_i^{\log} = a_i d_{\log}u + b_i d_{\log}v$ (and a_i or b_i is a local unit), so that, up to permuting u and v we can always assume that $a_1(\underline{\tilde{a}}) \neq 0$, thus $m \geq 1$.

Using again Equations (15) and (16) we obtain the following relations:

$$(24) \quad ux_u + uAy_u = u^m v^n \psi_1 a_1$$

$$(25) \quad uBx_u + uy_u = u^m v^n \psi_2 (u^{p-m} v^{q-n} a_2)$$

$$(26) \quad vx_v + vAy_v = u^m v^n \psi_1 b_1$$

$$(27) \quad vBx_v + vy_v = u^m v^n \psi_2 (u^{p-m} v^{q-n} b_2)$$

We deduce, up to changing v into $\text{Unit} \cdot v$, that $x = X_0(v) \pm u^m v^n$ and $y = Y_0(v) + u^m v^n Y(u, v)$. Since $m \geq 1$, Equations (26) and (27) implies that $X_0 = a_0 \in \mathbb{R}$ and $Y_0 = b_0 \in \mathbb{R}$. So we can write

$$Y = y_0 + vy_1(v) + uy_2(u) + uv y_3(u, v)$$

From Equation (26), we deduce that necessarily $b_1(0) \neq 0$, so that we have deduced

Lemma 10.10. *For any local coordinates (u, v) centered at the corner point \tilde{a} of \tilde{E} and adapted to \tilde{E} we find $\mu_1 = vdu + U\text{nit} \cdot u dv$.*

Proposition 10.11. *There exists local coordinates (u, v) centered at \tilde{a} adapted to \tilde{E} such that $(D\tilde{\pi})^*\Omega_{M_0}^1$ is locally generated at \tilde{a} as an $\mathcal{O}_{\tilde{X}}$ -sub-module of $\Omega_{\tilde{X}}^1(\log \tilde{E})$ by*

$$u^m v^n d_{\log}(u^m v^n) \text{ and } u^{r+p} v^{s+q} d_{\log}(u^{r+p} v^{s+q}).$$

Equivalently $(D\tilde{\pi})^\Omega_{M_0}^1$ is $\mathcal{O}_{\tilde{a}}$ -generated, as a submodule of $\Omega_{\tilde{X}, \tilde{a}}^1$ nearby \tilde{a} by $d(u^m v^n)$ and $d(u^{r+p} v^{s+q})$. Thus the plane vectors (m, n) and $(p+r, q+s)$ are linearly independent.*

Proof. We already have that $x = a_0 \pm u^m v^n$, the sign “ \pm ” may be “ $-$ ” only if m, n are both even.

We can write $y - b_0 = u^m v^n [f(u, v) + z(u, v)]$ for regular germ $f, g \in \mathcal{O}_{\tilde{a}}$ such that each monomial $u^k v^l$ appearing in f is such that $ml - kn = 0$, and each monomial $u^k v^l$ appearing in g is such that $ml - kn \neq 0$. Thus

$$dx \wedge dy = u^m v^n dx \wedge dz = U\text{nit} \cdot u^{r+p+m-1} v^{s+q+n-1} du \wedge dv.$$

Let $u^k v^l$ be a monomial of z , thus $dx \wedge d(u^k v^l) = (ml - nk)u^{k+m-1} v^{l+n-1}$. Necessarily we deduce that $z = u^{r+p} v^{s+q} \alpha$ for a local analytic unit α . This fact implies that necessarily the planes vectors (m, n) and $(p+r, q+s)$ are linearly independent. We are looking, if possible, for a change of local coordinates of the form $u = \bar{u}U$ and $v = \bar{v}V$ for local units U, V such that $\bar{u}^m \bar{v}^n = \pm u^m v^n$ and $u^{r+p} v^{s+q} \alpha = \pm \bar{u}^{r+p} \bar{v}^{s+q}$. Let ε be the sign of $\alpha(0)$. So we need $U^m V^n = 1$ and $U^{r+p} V^{s+q} = \varepsilon \alpha$ knowing that $m(s+q) - n(r+p) \neq 0$, this is equivalent to $V^{(s+q)n-(r+p)m} = (\varepsilon \alpha)^m$, which can be resolved. Thus we can re-write $x = a_0 \pm \bar{u}^m \bar{v}^n$ and $y = a_0 + \bar{u}^m \bar{v}^n [h(\bar{u}, \bar{v}) \pm \bar{u}^{r+p} \bar{v}^{s+q}]$ where h has only monomials $\bar{u}^k \bar{v}^l$ such that $ml - kn = 0$. These coordinates satisfy the announced result. \square

As in the smooth point case, using Hsiang & Pati proof, with a few elementary computations, we deduce from Proposition 10.11 the existence of Hsiang & Pati coordinates:

Corollary 10.12 (see also [2]). *There exist adapted coordinates (u, v) at the corner point \tilde{a} such that the resolution mapping $\tilde{\pi}$ locally writes*

$$(u, v) \rightarrow (x, y, z) = \tilde{\pi}(\tilde{a}) + (\pm u^m v^n, u^m v^n f(u, v) \pm u^k v^l; u^m v^n z(u, v) + u^k v^l Z(u, v)) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2}$$

for non-negative integers $k \geq m$ and $l \geq n$ such that $nk - lm \neq 0$ and function germs $f \in \mathcal{O}_{\tilde{a}}$ and $z, Z \in \mathcal{O}_{\tilde{a}}^{\dim M_0 - 2}$ such that $df \wedge d(u^m v^n) = dz \wedge d(u^m v^n) = 0$.

10.4. Local normal form for the induced metric.

Following on the material presented in the previous subsection, we will give local normal forms of the metric $g_0|_{X_0}$ pulled-back onto the resolved surface \tilde{X} at any point \tilde{a} of the exceptional divisor \tilde{E} according to being a regular point or a corner point.

From Theorem 9.2, Proposition 10.4, Proposition 10.7 and Proposition 10.11 there exist

$$U := \mathcal{M}_1^{\log} \text{ and } T := \mathcal{M}_2^{\log} \text{ and } V := T \cdot \mathcal{M}_{1,2}^{\log},$$

ordered monomials in \tilde{E} with $T = U \cdot (\dots)$ and $V = T \cdot (\dots)$ such that the pulled-back metric on \tilde{X} writes nearby \tilde{a} .

$$(D\tilde{\pi})^*g_0|_{X_0} = \lambda_1(\mathcal{M}_1 \mu_1) \otimes (\mathcal{M}_1 \mu_1) + \lambda_2(\mathcal{M}_2 \mu_2) \otimes (\mathcal{M}_2 \mu_2)$$

for positive analytic units λ_1, λ_2 . Moreover we know that $\{U\mu_1^{\log}, T\mu_2^{\log}\}$ and $\{U d_{\log} U, V d_{\log} V\}$ are both $\mathcal{O}_{\tilde{a}}$ -basis of the sub-module $(D\tilde{\pi})^*\Omega_{M_0}^1 \subset \Omega_{\tilde{X}}^1(\log E)$ nearby \tilde{a} . Thus we can write,

$$(28) \quad U\mu_1^{\log} = C_1 U d_{\log} U + D_1 V d_{\log} V$$

$$(29) \quad T\mu_2^{\log} = C_2 U d_{\log} U + D_2 V d_{\log} V$$

with $C_1D_2 - C_2D_1 = \text{Unit}$. Let us write $\eta_1 := d_{\log}U$ and $\eta_2 := d_{\log}V$. If $T \neq U$ in Equation (29), then $C_1D_2 = \text{Unit}$.

Thus we can write

$(D\tilde{\pi})^*g_0|_{X_0} = (\lambda_1C_1^2 + \lambda_2C_2^2)\eta_1 \otimes \eta_1 + (\lambda_1D_1^2 + \lambda_2D_2^2)\eta_2 \otimes \eta_2 + (\lambda_1C_1D_1 + \lambda_2C_2D_2)(\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1)$ for positive analytic units λ_1, λ_2 . Since $C_1D_2 - C_2D_1$ is a unit, as a quadratic form in the "variables" η_1, η_2 , the pulled-back metric $(D\tilde{\pi})^*\kappa$ is positive definite nearby \tilde{u} , thus we deduce the following looked for and expected

Proposition 10.13. *At the point \tilde{u} of \tilde{E} the pulled-back metric $(D\tilde{\pi})^*g_0|_{X_0}$ is locally quasi-isometric to the following metric:*

$$\begin{aligned} U^2\eta_1 \otimes \eta_1 + V^2\eta_2 \otimes \eta_2 &= dU \otimes dU + dV \otimes dV \\ &= d\mathcal{M}_1^{\log} \otimes d\mathcal{M}_1^{\log} + d(\mathcal{M}_2^{\log} \mathcal{M}_{1,2}^{\log}) \otimes d(\mathcal{M}_2^{\log} \mathcal{M}_{1,2}^{\log}) \end{aligned}$$

• When \tilde{u} is a regular point of \tilde{E} , we can be a little more specific. We recall that by Lemma 10.8, we can assume that for any given $p \geq s - r + t + 2$, the coordinates (u, v) are such that $\mu_1 = du + u^p c_1 dv$. As a consequence of Proposition 10.6 and Proposition 10.7 and of elementary computations we find

Proposition 10.14. *Let \tilde{u} be a regular point of \tilde{E} we obtain. For each integer number $\rho \geq 2s + 2t + 1$, there exists an local adapted coordinates (u, v) such that*

$$(D\tilde{\pi})^*g_0 = \lambda_1 u^{2r} du \otimes du + \lambda_2 u^{2s+2t} dv \otimes dv + u^\rho (\dots) (du \otimes dv + dv \otimes du)$$

for positive analytic units λ_1, λ_2 .

Proof. Let ρ be given. We can already assume that p is chosen so that $2r + p \geq \rho$. We write μ_2 as $B\mu_1 + u^t dv$ for $B \in \mathfrak{m}_{\tilde{u}}$ and $t \geq 1$. We can write

$$(D\tilde{\pi})^*g_0 = [\text{Unit}^2 \cdot u^{2r} + \text{Unit}^2 \cdot u^{2s+2t} B^2] \mu_1^2 + 2\text{Unit}^2 \cdot u^{2s+t} B \mu_1 dv + \text{Unit}^2 \cdot u^{2s+2t} (dv)^2$$

Let us consider a change of variable of the form $u = w(1 + w^q A(v))$ with q a positive integer. Then we deduce that

$$\begin{aligned} du &= [1 + (q+1)w^q A] dw + w^{q+1} A' dv \\ du \otimes dv &= \frac{1 + (q+1)w^q A}{2} (dw \otimes dv + dv \otimes dw) + w^q A' dv \otimes dv \\ (du)^2 &= [1 + (q+1)w^q A]^2 dw \otimes dw + w^{q+1} [1 + (q+1)w^q A] A' (dw \otimes dv + dv \otimes dw) \\ &\quad + w^{2q+2} (A')^2 dv \otimes dv \end{aligned}$$

Observe that $\mu_1^2 = (rdu)^2 + u^p (\dots)$. So that in the new coordinates we find

$$(D\tilde{\pi})^*g_0|_{X_0} = \text{Unit}^2 \cdot w^{2r} dw \otimes dw + w^r C (dv \otimes dw + dw \otimes dv) + w^{2r} (\dots) dv \otimes dv \text{ where}$$

Where $w^r C = (\lambda_1 w^{2r+q+1} [1 + (q+1)w^q A]^{2r+1} A' + \lambda_2 w^{2s+t} [1 + (q+1)w^q A]^{2s+t} [B + w^t (\dots)])$ Since $B(0, v) = v^l b_0(v)$ for a positive integer l and a local analytic unit $b_0(v)$, taking $q = 2s - 2r + t - 1$, we find $A(v) = v^{l+1} a_0(v)$ for a local analytic unit a_0 , resolving a differential equation in v of the form $A' \psi(A) = v^l f(v)$ where ψ and f are local analytic unit in v , such that $w^r C = w^{2s+t+1} (\dots)$. The metric then writes

$$(D\tilde{\pi})^*g_0|_{X_0} = \text{Unit}^2 \cdot w^{2r} dw \otimes dw + \text{Unit}^2 u^{2s+2} (u^{t-1} dv + G dw) \otimes (u^{t-1} dv + G dw) \text{ with } G \in \mathfrak{m}_{\tilde{u}}.$$

Up to factoring further powers of u from G , we may assume that $G(0, v) \neq 0$, which is the worse case scenario. Replacing μ_1 by dw and μ_2 by $Hdu + u^{t-1} dv$, we check that after at most $t - 1$ consecutive changes of coordinates in the *exceptional* variable u_{exc} , of the form

$$u_{exc, old} := u_{exc, new} [1 + u_{exc, new}^{q_{new}} A_{new}(v)],$$

we find adapted coordinates (x, v) , with $(\tilde{E}, \tilde{u}) = \{x = 0\}$, such that

$$(D\tilde{\pi})^* \mathbf{g}_0|_{X_0} = \text{Unit}^2 \cdot u^{2r} du \otimes du + \text{Unit}^2 \cdot u^{2s+2t} dv \otimes dv + x^{2s+2t+1}(\dots)(du \otimes dv + dv \otimes du).$$

From here, finitely many (iterated) changes of variables of the form $v = y + x^m J(y)$, for a positive integer m and regular function germ J to find, will provide the announced result. \square

• When \tilde{a} is a corner point of \tilde{E} , we know that the form μ_1 (attached to the "smallest" monomial in \tilde{E} , writes $\mu_1 = uv[d_{\log} u + (\lambda + A)d_{\log} v]$ for local adapted coordinates (u, v) at \tilde{a} such that $-\lambda \notin \mathbb{Q}_{\geq 0}$ and $A \in \mathbf{m}_{\tilde{a}}$. Since our result is general and we do not have explicit equations of the foliations we are dealing with, the desingularization of the foliation locally given by μ_1 will tell us very little about λ .

Remarkably the very special context we are working in gives the value of λ , for free, namely

Corollary 10.15. $U\mu_1^{\log} = U d_{\log} U + G_1 V d_{\log} V$, for $G_1 \in \mathcal{O}_{\tilde{a}}$, so that $\lambda = \frac{n}{m}$.

Proof. Let $\xi := mu\partial_u - nv\partial_v$ so that $d_{\log} U(\xi) \equiv 0$. Using Proposition 10.13 and evaluating both quasi-isometric metrics along the vector field ξ provides the result. \square

Although this is not so surprising to find this value linked to the combinatorics of the resolution mapping, it is of consequence for applications, especially for geodesics (as in [10]) nearby singularities. This rational number will appear somehow in the local form of the geodesic vector field. Since it is rather complicated to study as such, trying a linear model makes sense, but linearization is very sensitive to arithmetic properties of the linear part of a vector field, since resonances (among the eigen-values of the linear part) are obstruction to linearization.

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